Solutions to Homework #18

1. Saracino, Section 17, Problem 17.9:
Let $R = \{q \in \mathbb{Q} \,|\, q = a/b \text{ with } a, b \in \mathbb{Z} \text{ and } b \text{ is odd}\}$. Prove that R has a unique maximal ideal.

Proof. Let $I = \langle 2 \rangle = \{2a/b \,|\, a, b \in \mathbb{Z} \text{ with } b \text{ odd}\}$. We will show that I is the unique maximal ideal.
Ideal: Since R is a commutative ring, any principal ideal is actually an ideal. [Alternately, you can prove from scratch that I is nonempty, is closed under $+$ and $-$, and satisfies the ideal property.]
Maximal: Note that $I \neq R$ since $1/1 \in R \smallsetminus I$. That is, I is a *proper* ideal.
Given an ideal J with $I \subsetneq J \subseteq R$, there is some $x \in J \smallsetminus I$. Write $x = a/b$ with $a, b \in \mathbb{Z}$ and b odd; by definition of I, we must have a odd, since $x \notin I$. Thus, $x^{-1} = b/a \in R$.
Hence, given any $y \in R$, we have $y = (yx^{-1})x \in J$ since $yx^{-1} \in R$ and $x \in J$. Thus, $R \subseteq J$ and hence $J = R$, proving that I is maximal.
Unique. Suppose there were some maximal ideal $J \subsetneq R$ with $J \neq I$. If $J \subseteq I$, then $J \subsetneq I \subseteq R$, contradicting the maximality of J. Thus, $J \not\subseteq I$, and hence there is some $x \in J \smallsetminus I$.
We now follow the exact same argument as above: given any $y \in R$, we have $y = (yx^{-1})x \in J$ since $yx^{-1} \in R$ and $x \in J$. Thus, $R \subseteq J$ and hence $J = R$, contradicting the fact that J is maximal and therefore proper. By this contradiction, I is indeed unique as a maximal ideal. QED

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2. Saracino, Section 17, Problem 17.13:
Let I be an ideal of a ring R. Prove that the distributive laws hold in R/I.

Proof. Given $I + x, I + y, I + z \in R/I$, with $x, y, z \in R$, we have

$$\begin{aligned}(I+x)\big((I+y)+(I+z)\big) &= (I+x)\big(I+(y+z)\big) = I + x(y+z) = I + (xy+xz) \\ &= (I+xy)+(I+xz) = (I+x)(I+y)+(I+x)(I+z)\end{aligned}$$

and

$$\begin{aligned}\big((I+y)+(I+z)\big)(I+x) &= \big(I+(y+z)\big)(I+x) = I + (y+z)x = I + (yx+zx) \\ &= (I+yx)+(I+zx) = (I+y)(I+x)+(I+z)(I+x)\end{aligned}$$

QED

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3. Saracino, Section 17, Problem 17.14:
Let R be a ring and I an ideal of R.

(a) If R is commutative, prove that R/I is commutative.

(b) If R has unity, prove that R/I has unity.

Proof. (a) Given $I + x, I + y \in R/I$, with $x, y \in R$, we have

$$(I+x)(I+y) = I + xy = I + yx = (I+y)(I+x)$$

(b) Let 1_R denote the unity element of R. We claim that $I + 1_R$ is a unity of R/I. To see this, given $I + x \in R/I$, with $x \in R$, we have

$$(I+x)(I+1_R) = I + x1_R = I + x \quad \text{and} \quad (I+1_R)(I+x) = I + 1_Rx = I + x$$

QED

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4. Saracino, Section 17, Problem 17.22(a,b):
Let R be a commutative ring and X a subset of R. The **annihilator** of X is

$$\operatorname{Ann}(X) = \{r \in R \,|\, rx = 0 \text{ for every } x \in X\}.$$

(a) Prove that $\operatorname{Ann}(X)$ is an ideal of R.

(b) Let $R = \mathbb{Z}/12\mathbb{Z}$. Find $\operatorname{Ann}(\{2\})$.

Proof. (a) (**Nonempty**) We claim that $0 \in \operatorname{Ann}(X)$. To see this, given $x \in X$, we have $0x = 0$, as desired.

(**Closed**) Given $r, s \in \operatorname{Ann}(X)$, we claim that $r - s \in \operatorname{Ann}(X)$. To see this, given $x \in X$, we have

$$(r-s)x = rx - sx = 0 - 0 = 0, \text{ as desired.}$$

(**Sticky**) Given $r \in \operatorname{Ann}(X)$ and $y \in R$, we claim that $yr, ry \in \operatorname{Ann}(X)$. Since $ry = yr$ (because R is commutative), we need only prove one of these. Given $x \in X$, we have

$$(yr)x = y(rx) = y0 = 0, \text{ as desired.} \qquad \text{QED (a)}$$

(b): We claim that $\operatorname{Ann}(\{2\}) = \{0, 6\}$, as we now prove:

$(\subseteq)$: Given $n \in \operatorname{Ann}(\{2\})$, we have $2n = 0_{\mathbb{Z}/12\mathbb{Z}}$, i.e., $2n \equiv 0 \pmod{12}$. That is, n is an integer such that $2n$ is divisible by 12, so n must be divisible by 6. The only such integers in $\mathbb{Z}/12\mathbb{Z}$ are $n = 0, 6$.

$(\supseteq)$: We have $0 \cdot 2 = 0$ and $6 \cdot 2 = 0$ (modulo 12), so $0, 6 \in \operatorname{Ann}(\{2\})$. QED

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5. Saracino, Section 17, Problem 17.33:
Let R be a ring, and let I and J be ideals of R. Define $I + J = \{x + y \mid x \in I,\ y \in J\}$.

(a) Prove that $I + J$ is an ideal of R.

(b) Let $R = \mathbb{Z}$. Find $6\mathbb{Z} + 14\mathbb{Z}$.

Proof. (a): We have $0 \in I, J$, and hence $0 = 0 + 0 \in I + J$, so $I + J$ is nonempty.
Given $a, b \in I + J$, write $a = s + t$ and $b = x + y$ with $s, x \in I$ and $t, y \in J$. Then

$$a - b = s + t - (x + y) = (s - x) + (t - y) \in I + J.$$

Finally, given $a \in I + J$ and $r \in R$,write $a = x + y$ with $x \in I$ and $y \in J$. Then

$$ar = (x + y)r = xr + yr \in I + J,$$

and similarly $ra \in I + J$.

(b): [Note: every ideal of $\mathbb{Z}$ is principal, i.e., of the form $n\mathbb{Z}$. Since we know from part (a) that $6\mathbb{Z} + 14\mathbb{Z}$ is an ideal, we only need to find which integer n to use. It turns out it's $\gcd(6, 14) = 2$, in light of Theorem 4.2, that $2 = 6x + 14y$ for some integers x, y. E.g. $x = -2$, $y = 1$ work.]
We claim that $6\mathbb{Z} + 14\mathbb{Z} = 2\mathbb{Z}$. To prove $(\subseteq)$, given $6x + 14y \in$ LHS with $x, y \in \mathbb{Z}$, we have $6x + 14y = 2(3 + 7y) \in 2\mathbb{Z}$. To prove $(\supseteq)$, given $2n \in 2\mathbb{Z}$ with $n \in \mathbb{Z}$, we have $-12n \in 6\mathbb{Z}$ and $14n \in 14\mathbb{Z}$, and hence $2n = -12n + 14n \in$ LHS. QED

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6. Saracino, Section 18, Problem 18.1(b,c,e):
Which of the following are ring homomorphisms? [Prove your answers, of course]

(b) $\varphi : \mathbb{C} \to \mathbb{C}$ by $\varphi(a + bi) = a - bi$

(c) $\varphi : \mathbb{C} \to \mathbb{R}$ by $\varphi(a + bi) = a$

(e) Let R be the ring of polynomials with real coefficients, and let $\varphi : R \to R$ by $\varphi(p(x)) = p'(x)$, the derivative of $p(x)$.

Solution. (b): $\boxed{\text{YES, ring homomorphism}}$ Given $z, w \in \mathbb{C}$, write $z = a + bi$ and $w = c + di$ with $a, b, c, d \in \mathbb{R}$. Then

$$\varphi(z + w) = \varphi\big((a + c) + (b + d)i\big) = (a + c) - (b + d)i = (a - bi) + (c - di) = \varphi(z) + \varphi(w),$$

and

$$\varphi(zw) = \varphi\big((ac - bd) + (ad + bc)i\big) = (ac - bd) - (ad + bc)i = (a - bi)(c - di) = \varphi(z)\varphi(w). \qquad \text{QED (a)}$$

(c): $\boxed{\text{NO, not ring homomorphism}}$ Let $z = w = i \in \mathbb{C}$. Then $\varphi(z) = \varphi(w) = 0$. However, $zw = -1$, and $\varphi(-1) = -1$. Thus,

$$\varphi(zw) = \varphi(-1) = -1 \neq 0 = 0 \cdot 0 = \varphi(z) \cdot \varphi(w).$$

(e): $\boxed{\text{NO, not ring homomorphism}}$ Let $p(x) = q(x) = x$. Then $\varphi(p) = \varphi(q) = 1$. However, $pq(x) = x^2$, so $\varphi(pq) = 2x$. Thus,

$$\varphi(pq) = \varphi(x^2) = 2x \neq 1 = 1 \cdot 1 = \varphi(p) \cdot \varphi(q).$$