Solutions to Homework #14

1. Saracino, Section 11, Problem 11.20: Let G be a group, and let $H \lhd G$ be a normal subgroup such that $[G : H] = 20$ and $|H| = 7$. Suppose that $x \in G$ and $x^7 = e$. Prove that $x \in H$.

Proof. Since $H \lhd G$, we may consider the quotient group G/H.
Then $Hx \in G/H$, with $(Hx)^7 = H(x^7) = He$. Thus, the order of Hx (as an element of G/H) divides 7. On the other hand, since $|G/H| = [G : H] = 20$, the order $o(Hx)$ also divides 20. Since $\gcd(7, 20) = 1$, we have $o(Hx) = 1$, and hence $Hx = He$. Thus, $x = xe^{-1} \in H$. QED

[**Note**: We never used the hypothesis that $|H| = 7$. That was just a red herring.]

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2. Saracino, Section 11, Problem 11.30(b), slight variant:
Let G be a group. A **commutator** is an element of G that can be written as $xyx^{-1}y^{-1}$ for some $x, y \in G$. Let $C \subseteq G$ be the set of all the commutators, i.e., $C = \{xyx^{-1}y^{-1} \mid x, y \in G\}$.

Prove that for any subgroup $K \subseteq G$, the following are equivalent:
(i) $C \subseteq K$.
(ii) $K \lhd G$, and G/K is abelian.

Proof. ($\Rightarrow$): (Normality): Given $k \in K$ and $g \in G$, we have $gkg^{-1} = (gkg^{-1}k^{-1})k \in K$, since $gkg^{-1}k^{-1} \in C \subseteq K$, and $k \in K$, and K is closed under multiplication.
(Abelian): Given $Kx, Ky \in G/K$, i.e., given $x, y \in G$, we have $(xy)(yx)^{-1} = xyx^{-1}y^{-1} \in C \subseteq K$, and hence $Kxy = Kyx$, i.e., $(Kx)(Ky) = (Ky)(Kx)$.
($\Leftarrow$): Given $c \in C$, there exist $x, y \in G$ such that $c = xyx^{-1}y^{-1}$.
We have $(Kx)(Ky) = (Ky)(Kx)$ since G/K is abelian.
Then $Kxy = Kyx$, and hence $c = xyx^{-1}y^{-1} = (xy)(yx)^{-1} \in K$. QED

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3. Saracino, Section 12, Problem 12.1(a,b,c,d): Which of the following mappings are homomorphisms? For those that are, which are one-to-one, which are onto, and which are isomorphisms?
(a) $G = \mathbb{R}^\times$, $H = \mathbb{R}_{>0}$, and $\varphi : G \to H$ by $\varphi(x) = |x|$.
(b) $G = \mathbb{R}_{>0}$, and $\varphi : G \to G$ by $\varphi(x) = \sqrt{x}$.
(c) G= group of polynomials with real coefficients, under addition, and $\varphi : G \to \mathbb{R}$ by $\phi(p) = p(1)$.
(d) G as in (c), and $\phi : G \to G$ by $\phi(p) = p'$, the derivative of $p(x)$.

Answers/Proofs. (a): $\boxed{\text{Homomorphism and onto, but NOT one-to-one}}$

Homom: Given $x, y \in G$, we have $\varphi(xy) = |xy| = |x||y| = \varphi(x)\varphi(y)$.

Onto: Given $y \in H$, we have $y \in G$ since y is a nonzero real number, and because $y > 0$, we have $\varphi(y) = |y| = y$.

Not 1-1: Let $x_1 = 1$ and $x_2 = -1$. Then $x_1 \neq x_2$ but $\varphi(x_1) = 1 = \varphi(x_2)$. QED

(b): $\boxed{\text{Isomorphism}}$

Homom: Given $x, y \in G$, we have $\varphi(xy) = \sqrt{xy} = \sqrt{x}\sqrt{y} = \varphi(x)\varphi(y)$ since $x, y > 0$.

Onto: Given $y \in G$, let $x = y^2 \in G$. Then because $y > 0$, we have $\varphi(x) = \sqrt{y^2} = |y| = y$.

1-1: Given $x_1, x_2 \in G$ such that $\varphi(x_1) = \varphi(x_2)$, we have $\sqrt{x_1} = \sqrt{x_2}$, and hence, squaring both sides, $x_1 = x_2$. QED

(c): $\boxed{\text{Homomorphism and onto, but NOT one-to-one}}$

Homom: Given $p, q \in G$, we have $\varphi(p + q) = (p + q)(1) = p(1) + q(1) = \varphi(p) + \varphi(q)$.

Onto: Given $b \in \mathbb{R}$, let $p \in G$ be the constant polynomial $p(x) = b$. Then $\varphi(p) = p(1) = b$.

Not 1-1: Let $p_1(x) = 0$ and $p_2(x) = x - 1$. Then $p_1 \neq p_2$ but $\varphi(p_1) = p_1(1) = 0 = p_2(1) = \varphi(p_2)$. QED (d): $\boxed{\text{Homomorphism and onto, but NOT one-to-one}}$

Homom: Given $p, q \in G$, we have $\varphi(p+q) = (p+q)' = p' + q' = \varphi(p) + \varphi(q)$.

Onto: Given $q \in G$, let $p \in G$ be an antiderivative of $q(x)$, which we know exists from calculus. (It's always possible to antidifferentiate a polynomial.) Then $\varphi(p) = p' = q$.

Not 1-1: Let $p_1(x) = 0$ and $p_2(x) = 1$. Then $p_1 \neq p_2$ but $\varphi(p_1) = p_1' = 0 = p_2' = \varphi(p_2)$. QED

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4. Saracino, Section 12, Problem 12.3, first part: Let G be an abelian group, let $n \geq 1$ be a positive integer, and let $\varphi : G \to G$ by $\varphi(x) = x^n$. Prove that φ is a homomorphism.

Proof. Given $x, y \in G$, we have

$$\varphi(xy) = (xy)^n = x^n y^n = \varphi(x)\varphi(y)$$

where the second equality is because G is abelian. QED

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5. Saracino, Section 12, Problem 12.4(a,b): In each case, determine whether or not the two groups are isomorphic.

(a) $(C_{12}, \oplus)$ and $(\mathbb{Q}_{>0}, \cdot)$

(b) $(2\mathbb{Z}, +)$ and $(3\mathbb{Z}, +)$

Solutions. (a): C_{12} and $\mathbb{Q}_{>0}$ are $\boxed{\text{Not isomorphic}}$
This is because $|C_{12}| = 12 \neq \infty = |\mathbb{Q}_{>0}|$. QED

(b): $2\mathbb{Z}$ and $3\mathbb{Z}$ are $\boxed{\text{isomorphic}}$
Method 1: Note that $2\mathbb{Z} = \langle 2 \rangle$ and $3\mathbb{Z} = \langle 3 \rangle$ are both infinite cyclic groups. By Theorem 12.3, they are isomorphic to one another.
Method 2: Define $\varphi : 2\mathbb{Z} \to 3\mathbb{Z}$ by $\varphi(n) = 3n/2$.
Then φ is defined since each $n \in 2\mathbb{Z}$ is of the form $n = 2m$ for some $m \in \mathbb{Z}$, and hence $\varphi(n) = 3m \in 3\mathbb{Z}$. It is a homomorphism because for any $x, y \in 2\mathbb{Z}$, we have

$$\varphi(x+y) = 3(x+y)/2 = 3x/2 + 3y/2 = \varphi(x) + \varphi(y).$$

To show φ is 1-1: Given $a, b \in 2\mathbb{Z}$ with $\varphi(a) = \varphi(b)$, we have $3a/2 = 3b/2$, and hence $a = b$.

Finally, to show φ is onto: Given $y \in 3\mathbb{Z}$, write $y = 3m$ for some $m \in \mathbb{Z}$. Then $2m \in 2\mathbb{Z}$, and $\varphi(2m) = 3m = y$, proving that φ is onto. QED

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6. Saracino, Section 12, Problem 12.5: Let G, H be groups. Prove that $G \times H \cong H \times G$.

Proof. Define $\varphi : G \times H \to H \times G$ by $\varphi(g, h) = (h, g)$.

Homom: Given $(g, h), (x, y) \in G \times H$, we have

$$\varphi\big((g,h)(x,y)\big) = \varphi(gx, hy) = (hy, gx) = (h, g)(y, x) = \varphi(g, h)\varphi(x, y)$$

Onto: Given $(h, g) \in H \times G$, we have $(g, h) \in G \times H$, and $\varphi(g, h) = (h, g)$.

1-1: Given $(g, h), (x, y) \in G \times H$ such that $\varphi(g, h) = \varphi(x, y)$, we have $(h, g) = (y, x)$, and hence $h = y$ and $g = x$. Therefore, $(g, h) = (x, y)$. QED