Solutions to Homework #13

1. Saracino, Section 11, Problem 11.5: Let G be a group, let $H \subseteq G$ be a subgroup, and let $K \lhd G$. Prove that $H \cap K \lhd H$.

Proof. By Theorem 5.4(i), $H \cap K$ is a subgroup of G and hence (being a group that is a subset of H) is a subgroup of H.
Given $x \in H \cap K$ and $h \in H$, then $hxh^{-1} \in H$ since H is a subgroup and $h, x \in H$. Moreover, $hxh^{-1} \in K$ since $x \in K$ and $h \in G$ and $K \lhd G$. Thus, $hxh^{-1} \in H \cap K$. QED

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2. Saracino, Section 11, Problem 11.8: Let G be a group, let $N \lhd G$, and let $H \subseteq G$ be any subgroup of G. Define

$$NH = \{nh \,|\, n \in N \text{ and } h \in H\}.$$

Prove that NH is a subgroup of G.

Proof. **(Nonempty)**: We have $e \in N$ and $e \in H$, and hence $ee \in NH$. Thus, $NH \neq \varnothing$.
(Closed): Given $n_1h_1, n_2h_2 \in NH$, i.e., $n_1, n_2 \in N$ and $h_1, h_2 \in H$, we have

$$(n_1h_1)(n_2h_2) = \big(n_1(h_1n_2h_1^{-1})\big)(h_1h_2) \in NH,$$

since $h_1n_2h_1^{-1} \in N$ since $N \lhd G$, and therefore $n_1(h_1n_2h_1^{-1}) \in N$, while $h_1h_2 \in H$.
(Inverses): Given $nh \in NH$, i.e., $n \in N$ and $h \in H$, we have

$$(nh)^{-1} = h^{-1}n^{-1} = (h^{-1}n^{-1}h)h^{-1} \in NH,$$

since $h^{-1}n^{-1}h \in N$ because $n^{-1} \in N$, $h^{-1} \in G$, and $N \lhd G$. QED

[**Alternative Proof of Closure**: Since $N \lhd G$, we have $h_1N = Nh_1$, and hence $h_1n_2 \in h_1N = Nh_1$, meaning that there is some $n' \in N$ such that $h_1n_2 = n'h_1$. Thus, $n_1h_1n_2h_2 = (n_1n')(h_1h_2) \in NH$.]
[**Alternative Proof of Inverses**: Since $N \lhd G$, we have $h^{-1}N = Nh^{-1}$, and hence there is some $n' \in N$ such that $h^{-1}n^{-1} = n'h^{-1}$. Thus, $(nh)^{-1} = n'h^{-1} \in NH$.]

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3. Saracino, Section 11, Problem 11.11 (slight rephrasing): Which of the 10 subgroups of D_4 are normal, and which are not?

Answer/Proof. We know from Section 8 that there are 10 subgroups of D_4, which we now analyze for normality.

D_4: normal. Since it is the whole group, $D_4 \lhd D_4$ is automatically normal in itself.

$\langle f^2 \rangle$ and $\{e\}$: normal. From a previous homework problem (Saracino 8.14), we have $Z(D_4) = \langle f^2 \rangle$, which contains both of these subgroups. Thus, by Theorem 11.2, both of these subgroups are normal in D_4.

$\{e, f^2, g, f^2g\}$, $\langle f \rangle$, and $\{e, f^2, fg, f^3g\}$: normal. All of these subgroups have order 4 and hence index $8/4 = 2$ in D_4. By Theorem 11.3, then, they are all normal in D_4.

$\langle g \rangle$, $\langle f^2g \rangle$, $\langle fg \rangle$, $\langle f^3g \rangle$: **not** normal. Each of the four subgroups here is of the form $H_i = \{e, f^ig\}$ for $i = 0, 1, 2, 3$. To prove none is normal, for each i, we need to find $h \in H_i$ and $x \in D_4$ such that $xhx^{-1} \notin H_i$. Well, for each $i = 0, 1, 2, 3$, we have $f \in D_4$ and $f^ig \in H_i$, but

$$f(f^ig)f^{-1} = (ff^i)(gf^{-1}) = f^{i+1}(fg) = f^{i+2}g \notin H_i.$$

That is, D_4 has 6 normal subgroups, namely D_4, $\{e, f^2, g, f^2g\}$, $\langle f \rangle$, $\{e, f^2, fg, f^3g\}$, $\langle f^2 \rangle$, and $\{e\}$.

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4. Saracino, Section 11, Problem 11.12(b): Let $G = A_4$. Show that there exists subgroups $H, K \subseteq G$ such that $K \lhd H$ and $H \lhd G$, but K is not normal in G.

Proof. Let $H = \{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}$, which is a normal subgroup of A_4 by Example 2, page 102.
Let $f = (1\,2)(3\,4) \in H$, and let $K = \{e, f\} = \langle f \rangle$, which is a subgroup of H of index $4/2 = 2$, and hence which is normal in H by Theorem 11.3.

However, $g = (1\,2\,3) \in A_4$ has $gfg^{-1} = (1\,2\,3)(1\,2)(3\,4)(1\,3\,2) = (1\,4)(2\,3) \notin K$, and hence $K \ntriangleleft A_4$. QED

[**Note 1 from RLB**: An alternative way to see that $K \triangleleft H$ is to observe that H is abelian, so all of its subgroups are normal in H.]

[**Note 2 from RLB**: As this example shows, "H is normal" is an imprecise statement; it should really be "H is normal in G", unless the "in G" part is abundantly clear from context. That is, "normal" is a property of the relationship between **two** groups, not just an property of the smaller group. As we see here, K is a subgroup of both H and G, but although K is normal in H, it is **not** normal in G.]

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5. Saracino, Section 11, Problem 11.27: Let H be a subgroup of G. Define

$$N(H) = \{g \in G \,|\, gHg^{-1} = H\},$$

which is called the *normalizer* of H in G.

(a) Prove that $N(H)$ is a subgroup of G.
(b) Prove that $H \triangleleft N(H)$.
(c) Let $K \subseteq G$ be a subgroup such that $H \triangleleft K$. Prove that $K \subseteq N(H)$.

Proof. (a): **Nonempty**: We have $e \in H$ because $e \in G$ and $eHe^{-1} = H$.

Closed: Given $x, y \in N(H)$, we have $xy \in G$ and

$$(xy)H(xy)^{-1} = (xy)H\big(y^{-1}x^{-1}\big) = x\big(yHy^{-1}\big)x^{-1} = xHx^{-1} = H,$$

where the last two equalities are because $x, y \in N(H)$. Thus, $xy \in N(H)$.

Inverses: Given $x \in N(H)$, we have $x^{-1} \in G$ and

$$x^{-1}H(x^{-1})^{-1} = x^{-1}Hx = x^{-1}\big(xHx^{-1}\big)x = (x^{-1}x)H(x^{-1}x) = eHe = H,$$

where in the second equality we replaced H by xHx^{-1}; these two sets are equal because $x \in N(H)$. Thus, $x^{-1} \in H$. QED

(b): First, we claim that H is a sub**set** of $N(H)$. Given $h \in H$, we have $hHh^{-1} = hH = H$, proving this claim. [The first equality is by the right coset relation $Hh^{-1} = He = H$, for example, and the second is similarly because $hH = eH = H$.] Thus, H is a subgroup of $N(H)$.

To prove normality, given $h \in H$ and $x \in N(H)$, we have $xhx^{-1} \in xHx^{-1} = H$. QED

(c): Given $k \in K$, we must show that $kHk^{-1} = H$. However, since $H \triangleleft K$, the (i)$\Rightarrow$(ii) part of Theorem 11.1 says this is true, and we are done. QED

Note from RLB: More Careful Proofs: It is, arguably, a little sketchy to push around set equalities like

$$(xy)H\big(y^{-1}x^{-1}\big) = x\big(yHy^{-1}\big)x^{-1} = xHx^{-1} = H$$

as I did in the "Closed" step of part (a) above. Instead, then, one can prove the various set equalities in the above proofs the usual way: by proving each set is contained in the other.

I won't do all of the relevant such proofs here, but for example, here is a more careful proof that $(xy)H\big(y^{-1}x^{-1}\big) = H$, assuming that $x, y \in N(H)$.

Proof of $(\subseteq)$: Given $g \in (xy)H\big(y^{-1}x^{-1}\big)$, there exists $h \in H$ such that $g = xyhy^{-1}x^{-1}$. But then, since $y \in N(H)$, we have $yhy^{-1} \in H$, and therefore $g = x(yhy^{-1})x^{-1} \in xHx^{-1} = H$. QED $(\subseteq)$

Proof of $(\supseteq)$: Given $h \in H$, then since $x \in N(H)$, we have $h \in H = xHx^{-1}$. That is, there exists $h' \in H$ such that $h = xh'x^{-1}$.

Similarly, since $y \in N(H)$, we have $h' \in H = yHy^{-1}$, so there exists $h'' \in H$ such that $h' = yh''y^{-1}$.

Thus, $h = xh'x^{-1} = xyh''y^{-1}x^{-1} \in (xy)H\big(y^{-1}x^{-1}\big)$. QED $(\supseteq)$

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6. Saracino, Section 11, Problem 11.10:

Let G be a group, let $g \in G$ have finite order m, and let $H \triangleleft G$. Prove that the order of the element $Hg \in G/H$ is finite and divides m.

Proof. We have $(Hg)^m = H(g^m) = He$, where the first equality is by the definition of the group law on G/H, and the second is because $g^m = e$, since $o(g) = m$. Thus, by Theorem 4.4(ii), we have $o(Hg)$ is finite and divides m. QED

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7. Saracino, Section 11, Problem 11.18: Let G be cyclic and let $H \subseteq G$ be a subgroup. Prove that G/H is cyclic.

Proof. Let x be a generator for G. It suffices to show that Hx is a generator for G/H.
Given an arbitrary element Hg in G/H, we have $g \in G$, and hence, since $G = \langle x \rangle$, there is some $n \in \mathbb{Z}$ such that $g = x^n$. Therefore, by the group law on G/H, we have $(Hx)^n = H(x^n) = Hg$. QED