Solutions to Midterm Exam 2

1. **(20 points)** Consider the group S_8 of permutations of the set $\{1, 2, 3, \dots, 8\}$. Let $\sigma, \tau \in S_8$ be the permutations

$$\sigma = (1, 2, 3)(4, 5, 6) \quad \text{and} \quad \tau = (1, 3, 8, 6)(2, 5)(4, 7)$$

(a) Write $\sigma\tau$ as a product of **disjoint** cycles.

(b) Compute the **order** of each of σ, τ, and $\sigma\tau$.

(c) Decide whether each of σ, τ, and $\sigma\tau$ is an **even** or **odd** permutation.

(d) Find an element of S_8 of order 15.

Solutions (a): $\sigma\tau = (1, 2, 3)(4, 5, 6)(1, 3, 8, 6)(2, 5)(4, 7) = \boxed{(2, 6)(3, 8, 4, 7, 5)}$

(b): $o(\sigma) = \text{lcm}(3, 3) = \boxed{3}$
$o(\tau) = \text{lcm}(4, 2, 2) = \boxed{4}$
$o(\sigma\tau) = \text{lcm}(2, 5) = \boxed{10}$

(c): σ is even + even = $\boxed{\text{even}}$
τ is odd + odd + odd = $\boxed{\text{odd}}$
$\sigma\tau$ is odd + even = $\boxed{\text{odd}}$

(d): Let $\lambda = \boxed{(1,2,3)(4,5,6,7,8)} \in S_8$. Then $o(\lambda) = \text{lcm}(3, 5) = 15$.

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2. **(15 points)** Let G be a group, and let $H, K \subseteq G$ be subgroups. Define a relation R on G by

$$a\,R\,b \quad \text{means} \quad \exists h \in H \text{ and } \exists k \in K \text{ such that } a = hbk.$$

Prove that R is an equivalence relation on G.

Proof. **Reflexive**: Given $a \in G$, we have $e \in H$ and $e \in K$, and $a = eae$, so $a\,R\,a$.

Symmetric: Given $a, b \in G$ with $a\,R\,b$, there exist $h \in H$ and $k \in K$ with $a = hbk$. Thus, $b = h^{-1}ak^{-1}$. Since $h^{-1} \in H$ and $k^{-1} \in K$, we have $b\,R\,a$.

Transitive: Given $a, b, c \in G$ with $a\,R\,b$ and $b\,R\,c$, there exist $h, x \in H$ and $k, y \in K$ with $a = hbk$ and $b = xcy$.
Thus, $a = hbk = h(xcy)k = (hx)c(yk)$. Since $hx \in H$ and $yk \in K$, we have $a\,R\,c$.

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3. **(15 points)** Let G be a group, let $H, K \subseteq G$ be subgroups, and let $a \in K$. Suppose that

$$|G| = 120, \qquad |H| = 30, \qquad H \subseteq K, \quad \text{and} \quad o(a) = 8$$

Prove that $K = G$.

Proof. Let $m = |K|$. Since H is a subgroup of G and also contained in K, it is a subgroup of K as well. Therefore, by Lagrange's Theorem, we have $30|m$.
Since $x \in K$, we also have $o(x)|m$ by a corollary of Lagrange. That is, $8|m$.
Thus, m is divisible by $\text{lcm}(30, 8) = 120$. But $1 \le m \le 120$ since $K \subseteq G$, so $m = 120$. That is, K is a 120-element subset of the 120-element set G, so we have $K = G$. QED

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4. **(15 points)** Let G and H be groups, and let $\varphi : G \to H$ and $\psi : G \to H$ be homomorphisms. Define

$$K = \{x \in G \mid \varphi(x) = \psi(x)\}$$

Prove that K is a subgroup of G.

Proof. (**Nonempty**): We have $\varphi(e_G) = e_H = \psi(e_G)$, so $e_G \in K$. Thus, $K \neq \varnothing$.

(**Closed**): Given $x, y \in K$, we have

$$\varphi(xy) = \varphi(x)\varphi(y) = \psi(x)\psi(y) = \psi(xy),$$

and hence $xy \in K$.

(**Inverses**): Given $x \in K$, we have

$$\varphi(x^{-1}) = \varphi(x)^{-1} = \psi(x)^{-1} = \psi(x^{-1}),$$

and hence $x^{-1} \in K$. QED

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5. **(20 points)** Let G be a group, and let $H \subseteq G$ be a subgroup with the property that

$$xyx^{-1}y^{-1} \in H \quad \text{for all } x, y \in G.$$

Prove that $H \lhd G$, and that G/H is abelian.

Proof. (**Normal**): We already know H is a subgroup of G. Given $h \in H$ and $g \in G$, we have

$$ghg^{-1} = (ghg^{-1}h^{-1})h \in H,$$

since $ghg^{-1}h^{-1} \in H$ by hypothesis.

(**Abelian**): Given $Hx, Hy \in G/H$ (i.e., given $x, y \in G$), we have $xy(yx)^{-1} = xyx^{-1}y^{-1} \in H$, and therefore

$$HxHy = Hxy = Hyx = HyHx$$

QED

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6. **(15 points)** Let G be a group, and let $H \lhd G$ be a normal subgroup of index $[G : H] = 30$.
Prove that $x^{30} \in H$ for every $x \in G$.

Proof. Given $x \in G$, consider $Hx \in G/H$, which is a group because $H \lhd G$. We have $|G/H| = 30$, so by a corollary to Lagrange, we have $(Hx)^{30} = He$. That is, $Hx^{30} = He$, or equivalently, $x^{30}e^{-1} \in H$, which means $x^{30} \in H$. QED

OPTIONAL BONUS. (2 points.) Let G be a group; recall that the center $Z(G)$ is the normal subgroup consisting of all elements of G that commute with every element of G.
Suppose that $G/Z(G)$ is cyclic. Prove that G is abelian.

Proof. By hypothesis, there exists $a \in G$ such that the coset $Z(G)a$ is a generator for the (cyclic) group $G/Z(G)$.

Given $x, y \in G$, consider the cosets $Z(G)x$ and $Z(G)y$, which are elements of $G/Z(G)$. Thus, there are integers $m, n \in \mathbb{Z}$ such that $Z(G)x = (Z(G)a)^m$ and $Z(G)y = (Z(G)a)^n$.

That is, we have $xa^{-m} \in Z(G)$ and $ya^{-n} \in Z(G)$. Let $w = xa^{-m} \in Z(G)$, and let $z = ya^{-n} \in Z(G)$. Then $x = wa^m$ and $y = za^n$. Therefore,

$$xy = wa^m za^n = zwa^m a^n = zwa^{m+n} = zwa^n a^m = za^n wa^m = yx,$$

where the second equality is because z commutes with everything, the third and fourth are by rules of exponents, and the fifth is because w commutes with everything. QED