Solutions to Midterm Exam 2

1. **(20 points)** Consider the group S_8 of permutations of the set $\{1, 2, 3, \dots, 8\}$. Let $\sigma, \tau \in S_8$ be the permutations

$$\sigma = (1, 2, 3)(4, 5, 6) \quad \text{and} \quad \tau = (1, 3, 8, 6)(2, 5)(4, 7)$$

(a) Write $\sigma\tau$ as a product of **disjoint** cycles.

(b) Compute the **order** of each of σ, τ, and $\sigma\tau$.

(c) Decide whether each of σ, τ, and $\sigma\tau$ is an **even** or **odd** permutation.

(d) Find an element of S_8 of order 15.

Solutions (a): $\sigma\tau = (1, 2, 3)(4, 5, 6)(1, 3, 8, 6)(2, 5)(4, 7) = \boxed{(2, 6)(3, 8, 4, 7, 5)}$

(b): $o(\sigma) = \text{lcm}(3, 3) = \boxed{3}$
$o(\tau) = \text{lcm}(4, 2, 2) = \boxed{4}$
$o(\sigma\tau) = \text{lcm}(2, 5) = \boxed{10}$

(c): σ is even + even = $\boxed{\text{even}}$
τ is odd + odd + odd = $\boxed{\text{odd}}$
$\sigma\tau$ is odd + even = $\boxed{\text{odd}}$

(d): Let $\lambda = \boxed{(1,2,3)(4,5,6,7,8)} \in S_8$. Then $o(\lambda) = \text{lcm}(3, 5) = 15$.

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2. **(15 points)** Let G be a group, and let $H, K \subseteq G$ be subgroups. Define a relation R on G by

$$a\,R\,b \quad \text{means} \quad \exists h \in H \text{ and } \exists k \in K \text{ such that } a = hbk.$$

Prove that R is an equivalence relation on G.

Proof. **Reflexive**: Given $a \in G$, we have $e \in H$ and $e \in K$, and $a = eae$, so $a\,R\,a$.

Symmetric: Given $a, b \in G$ with $a\,R\,b$, there exist $h \in H$ and $k \in K$ with $a = hbk$. Thus, $b = h^{-1}ak^{-1}$. Since $h^{-1} \in H$ and $k^{-1} \in K$, we have $b\,R\,a$.

Transitive: Given $a, b, c \in G$ with $a\,R\,b$ and $b\,R\,c$, there exist $h, x \in H$ and $k, y \in K$ with $a = hbk$ and $b = xcy$.
Thus, $a = hbk = h(xcy)k = (hx)c(yk)$. Since $hx \in H$ and $yk \in K$, we have $a\,R\,c$.

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3. **(15 points)** Let G be a group, and let $H, K \subseteq G$ be subgroups; as you know (and may assume without proof), $H \cap K$ is therefore also a subgroup.
Suppose that $|H| = 40$ and $|K| = 75$. Prove that $H \cap K$ is cyclic.

Proof. Let $m = |H \cap K|$. By an old theorem, $H \cap K$ is a subgroup of G, and therefore, being a subset of both H and K, it is also a subgroup of both H and K.
Applying Lagrange's Theorem to $H \cap K \subseteq H$, we have $m|40$, and applying to $H \cap K \subseteq K$ gives $m|75$.
Thus, m divides the gcd $(40, 75) = 5$. That is, either $m = 1$ or $m = 5$.
If $m = 1$, then $H \cap K = \{e\} = \langle e \rangle$ is cyclic.
If $m = 5$, then because 5 is prime, a corollary of Lagrange says that $H \cap K$ is cyclic. QED

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4. **(15 points)** Let G and H be groups, and let $\varphi : G \to H$ be an **onto** homomorphism. Prove that H is abelian **if and only if**

$$\varphi\big(xyx^{-1}y^{-1}\big) = e_H \quad \text{for all } x, y \in G.$$

Proof. $(\Rightarrow)$: Given $x, y \in G$, we have

$$\varphi(xyx^{-1}y^{-1}) = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = \varphi(x)\varphi(x)^{-1}\varphi(y)\varphi(y)^{-1} = e_H e_H = e_H,$$

where the second equality is because H is abelian.

($\Leftarrow$): Given $a, b \in H$, then because φ is onto, there exist $x, y \in G$ such that $\varphi(x) = a$ and $\varphi(y) = b$. Hence,

$$aba^{-1}b^{-1} = \varphi(x)\varphi(y)\varphi(x)^{-1}\varphi(y)^{-1} = \varphi(xyx^{-1}y^{-1}) = e_H.$$

Rearranging, then, we have $ab = ba$. QED

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5. **(20 points)** Let G be a group, let $K \subseteq G$ be a subgroup, and let $N \lhd G$ be a normal subgroup of G. Define

$$H = \{xy \,|\, x \in K \text{ and } y \in N\}.$$

Prove that H is a subgroup of G.

Proof. (**Nonempty**): We have $e_G \in N, K$, so $e_G e_G \in H$, so $H \neq \varnothing$.

(**Closed**): Given $x_1y_1 \in H$ and $x_2y_2 \in H$ (with $x_i \in K$ and $y_i \in N$), then because $N \lhd G$, we have $Nx_2 = x_2N$, and hence there exists $y_3 \in N$ such that $y_1x_2 = x_2y_3$. Thus,

$$(x_1y_1)(x_2y_2) = x_1(y_1x_2)y_2 = x_1(x_2y_3)y_2 = (x_1x_2)(y_3y_2) \in H,$$

since $x_1x_2 \in K$ and $y_3y_2 \in N$.

(**Inverses**): Given $xy \in H$ (with $x \in K$ and $y \in N$), then because $N \lhd G$, we have $xN = Nx$, and hence there exists $y_1 \in N$ such that $xy = y_1x$. Thus,

$$(xy)^{-1} = (y_1x)^{-1} = x^{-1}y_1^{-1} \in H$$

since $x^{-1} \in K$ and $y_1^{-1} \in N$.

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6. **(15 points)** Let G be a group, and let $H \lhd G$ be a normal subgroup of index $[G : H] = 18$.
Prove that $x^{18} \in H$ for every $x \in G$.

Proof. Given $x \in G$, consider $Hx \in G/H$, which is a group because $H \lhd G$. We have $|G/H| = 18$, so by a corollary to Lagrange, we have $(Hx)^{18} = He$. That is, $Hx^{18} = He$, or equivalently, $x^{18}e^{-1} \in H$, which means $x^{18} \in H$. QED

OPTIONAL BONUS. (2 points.) Recall that D_n denotes the dihedral group of order $2n$, the symmetries of a regular n-gon.
Let $\varphi : D_{2021} \to C_{2021}$ be a homomorphism. Prove that φ is trivial. That is, prove that $\varphi(x) = 0$ for all $x \in D_{2021}$.

Proof. Let $y = \varphi(g) \in C_{2021}$. Then $2y = 2\varphi(g) = \varphi(g^2) = \varphi(e) = 0$, since φ is a homomorphism. Hence, $o(y)$ is either 1 or 2. However, by a corollary to Lagrange, we have $o(y)|2021$, since $|C_{2021}| = 2021$. Thus, $o(y) \neq 2$. So we must have $o(y) = 1$, and therefore $y = 0$. That is, we have shown that $\varphi(g) = 0$.
Since $(fg)^2 = e$ as well, we similarly have $\varphi(fg) = 0$. Thus,

$$\varphi(f) = \varphi(fgg) = \varphi(fg) + \varphi(g) = 0 + 0 = 0.$$

Given any $x \in D_{2021}$, we may write $x = f^ig^j$ for some integers i, j. Then

$$\varphi(x) = i\varphi(f) + j\varphi(g) = i \cdot 0 + j \cdot 0 = 0.$$

QED