Math 350, Sections 01,03, Fall 2021

Review Sheet for Exam 2

The exam (in class, Friday, November 12) will cover Sections 0–12 of Saracino, along with any related material from class, though of course the emphasis will be on the topics covered after Exam 1. The following is a list of most of those topics, spanning Sections 8–12. **THIS IS NOT A COMPREHENSIVE LIST, BUT MERELY AN AID.** As on Exam 1, when I list a given concept or definition below, I mean three things. First, you should know the official definition. Second, you should have a decent intuition for it. Third, you should also be able to use that concept in a proof.

You may bring one standard size (8.5x11") "cheat sheet" of notes to the exam

- Section 7: Besides the topics here listed on the review sheet for Exam 1, also know the following: Theorem 7.1: The set S_X of all invertible functions from X to X forms a group under composition (if $X \neq \varnothing$); it's called the symmetric group on X.

  What it means for a function to be *well-defined.* (This property is included in being a function, but it did not really come up until we got to Section 10 or so.)

- Section 8: Be able to work with elements of the group S_n and with the group operation. We won't use the (unnamed) 2-row matrix-style notation given on pages 66–67 very much; the (disjoint) cycle notation is much more efficient. Be able to work with permutations written in cycle notation, including being able to simplify the composition of two or more permutations. It's also useful for other computations: Given $\sigma \in S_n$, if we write it in cycle notation, then:

  - the order $o(\sigma)$ is the lcm of the lengths of the cycles in its disjoint cycle notation.
  - to decide whether σ is an even or odd permutation, remember that even-length cycles are odd, and odd-length cycles are even. So from its cycle notation, we can immediately tell that σ is odd if it has an odd number of even-length cycles; and it's even if it has an even number of even-length cycles.

  S_n is generated by the transpositions. (And know more generally what it means to say that a subset S generates a group G.) Know what "even" and "odd" mean; and know Theorem 8.4, that every permutation is even or odd but not both.

  A_n, the set of even permutations in S_n, is a subgroup, called the *alternating group* (of degree n, or on n letters). $|S_n| = n!$ and (for $n \geq 2$) $|A_n| = n!/2$.

  The *dihedral group* D_n of order $2n$ (or of degree n, or of symmetries of a regular n-gon), is also a subgroup of S_n. See the discussion about D_4 on pages 75–76, and the more general discussion of D_n in Exercise 8.15 and from the videos and from class.

- Section 9: Equivalence relation (on a set S); reflexive, symmetric, and transitive. A partition on S. Know how each one gives the other (i.e., Theorem 9.1). Equivalence classes.

  Given a group G and a subgroup H: the right coset equivalence relation $\equiv_H$, and the right cosets Ha; the right cosets are the equivalence classes of $\equiv_H$. The eight equivalent formulations of $x \equiv_H y$ from the in-class Corollary, especially $\boxed{Hx = Hy \Leftrightarrow xy^{-1} \in H}$

  Same for left cosets, especially $\boxed{xH = yH \Leftrightarrow x^{-1}y \in H}$

  Right coset representatives, i.e., equivalence class representatives for the right coset relation. (And same for left cosets and the left coset relation.)

  Remember, most of the time, right cosets and left cosets are *not* the same things.

- Section 10: Know what it means to say two sets S, T have the same cardinality, i.e. $|S| = |T|$. (Namely, there is a one-to-one and onto function from S to T.) For any subgroup H of a group G and any element $a \in G$, we have $|Ha| = |aH| = |H|$. (Also know the simple one-to-one and onto function that takes H to the right coset Ha, and the one that takes H to the left coset aH.) Theorem 10.3: There are the same number of right cosets as left cosets.

  The index $[G : H]$ of the subgroup H in the group G. (Warning: $[G : H]$ is *not* defined to be $|G|/|H|$, since that fraction doesn't make sense if, say, both groups are infinite. Of course, it is *true* that the index equals $|G|/|H|$ *if* both groups are finite; but that is a *theorem*, not the definition.)

  Lagrange's Theorem and some of the results that follow from it, like Theorems 10.4 and 10.5.

  Recall what center $Z(G)$ and centralizer $Z(a)$ mean; both are subgroups. The equivalence relation of conjugacy on a group G; the equivalence classes are called *conjugacy classes*. Know the conjugacy classes of S_n. The conjugacy classes of an abelian group are just singletons. A given conjugacy class is a singleton if and only if that element of G is in $Z(G)$.

- Section 11: Normal subgroups; know the various equivalent definitions in Theorem 11.1. The notation $H \lhd G$. Be familiar with some examples of normal and non-normal subgroups. Theorems 11.2–11.4 (and Corollary 11.5) giving various conditions for normality.

  The definition of the quotient group G/H (including the group law, of course); know why it only makes sense if H is normal in G.

- Section 12: Homomorphisms and isomorphisms of groups. Their basic properties (e.g. Theorems 12.1, 12.4, and 12.5). Know what it means to say two groups are isomorphic, the notation $G \cong H$, and that this is an equivalence relation. Every cyclic group is isomorphic to either C_n or $\mathbb{Z}$ (Theorems 12.2 and 12.3). The image and inverse image subgroups associated with a homomorphism $\varphi : G \to H$; know their basic properties (Theorem 12.6).

Some things you don't need to know

- From class: The rotation groups of solid objects.
- Section 10: Fermat's Theorem and Euler's Theorem. These concern (for $n \geq 2$) the group I called U_n (and which the book calls $\mathbb{Z}_p \setminus \{0\}$ if $n = p$ is prime) The Euler ϕ-function, i.e. $\phi(n) = |U_n|$. (But you **will** need to know this stuff for the final.)
- Section 10: Lemma 10.8, about the size of a conjugacy class. The Class Equation (Theorem 10.9). (But you **will** need to know this stuff for the final.)
- Section 11: Cauchy's Theorem (and Theorem 11.7, which is Cauchy for abelian groups).
- Section 12: The specific words monomorphism, epimorphism, and automorphism. (But you **do** need to know what it means for a homomorphism to be one-to-one or onto.)
- Section 12: Cayley's Theorem (Theorem 12.7).

Tips

- All the tips from the Exam 1 review sheet still apply.
- Know the coset equivalence relations inside out, especially that $\boxed{Ha = Hb \Longleftrightarrow ab^{-1} \in H}$
- Equivalence relations and cosets and quotient groups are notoriously confusing for many students. Practice them a lot. Enough that you **really** understand what each is and how they relate to one another.
- Know Lagrange's Theorem and the results that follow from it well. You need to notice **instantly**, for example, that a group of order 42 cannot have an element of order 12, because $12 \nmid 42$.
- Know quotient groups. Remember, the elements are themselves *sets*. (Specifically, cosets.)
- If you want to **prove** that a giving subgroup $H \subseteq G$ is normal, either use the $gHg^{-1} \subseteq H$ (for all $g \in G$) definition, or else use a result like any of Theorems 11.2–11.4.

  On the other hand, if you want to **use** the fact that $H \lhd G$, then either use one of the equivalent (but seemingly stronger) conditions (ii) or (iii) in Theorem 11.1, **or** use the fact that the quotient group G/H **is a group**. For example, Exercises 11.10 and 11.20 (pages 106–107) are really hard if you try to use the definition of normal or Theorem 11.1, but they are MUCH easier if you work with the group G/H and apply things like Lagrange's Theorem to that group.
- Don't confuse the notation $\varphi^{-1}(H')$, the inverse image of the set H', with φ^{-1}, the inverse function of φ. In particular, the inverse function φ^{-1} only exists at all if φ is one-to-one and onto. However, for *any* function $\varphi : G \to H$ and *any* subset $H' \subseteq H$, the inverse image $\varphi^{-1}(H')$ is *always* defined; it is the subset of G consisting of elements that map into H'.

Some practice problems

If you're looking for **more practice**, here are some **totally optional** problems you may attempt. Some were from the homework, but some were not. Most of the higher-numbered problems are harder than anything I would put on a timed exam, but they are still good practice. It would probably be best to read most or all the problems but pick and choose just some of problems to try, rather than simply doing all of them in order.

The unnumbered problems, as well as the numbered problems in **bold face**, are computational or are definition practice. The rest are proof/theoretical problems. Problems in *italics* have very short proofs, although it may be hard to find the short proof. Problems with asterisks (*) are trickier.

- Find the order and the parity (even or odd) of each of the following elements of S_8:

  (a): $\sigma = (1,4,3)(3,5)(2,7,5)(1,6,2,4,7)$

  (b): $\sigma = (3,6,4)(1,5,2,4)(1,6,5,3,2)$

  (c): σ, τ, and $\sigma\tau$, where $\sigma = (1,2,3)(4,5,6)$ and $\tau = (2,7,8,5)(3,4)$
- In each part (a), (b), (c) of the previous problem, and for each $k = 1,2,3,4,5,6$, find the parity (even or odd) of $g_k\sigma$ and $f_k\sigma$, where $f_k = (7,k)$ and $g_k = (7,k,8)$.
- Section 8, #24, 25*
- Section 9, #**7**, 13, *14**
- Section 10, #**2(a), 3(a)**, 7, 9, 24, **26**, 27*
- Section 11, #3, 4, 7, 9, 13, **14(a), 14(b)***, 21, *23*, 28, 29*
- Section 12, #2, **4(e,k*)**, 7, **8**, 14, 20(a)