Homework #20

Due **Friday, May 2** in Gradescope by **11:59 pm ET**

READ $\boxed{\text{Section 19 in Saracino}}$

WATCH 1. $\boxed{\text{Optional: Video 39: Reduction of Polynomials}}$ (8:29)

2. $\boxed{\text{Optional: Video 40: Proof of Eisenstein}}$ (24:37)

3. $\boxed{\text{Required: Video 41: Maximal and Irreducible}}$ (12:53)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (20 points) Saracino, Section 19, Problem 19.2(a,b,c)
For each of the following polynomials, determine whether or not they are irreducible in $\mathbb{Q}[X]$. [As always, justify your answers.]

(a) $X^3 + X + 36$

(b) $2X^3 - 8X^2 - 6X + 20$

(c) $2X^4 + 3X^3 + 15X + 6$

Problem 2. (15 points) Saracino, Section 19, Problem 19.3(a,d)
Write each of the following polynomials as a product of irreducible polynomials over the given field. (That is, factor each polynomial over the given field.)

(a) $2X^3 + X^2 + 2$ over $\mathbb{F}_3$

(d) $X^4 + X^3 + 2X^2 + X + 2$ over $\mathbb{F}_3$

Problem 3. (15 points) Saracino, Section 19, Problem 19.12
Let R be a commutative ring, let $r \in R$, and let $f, g \in R[X]$. Define $h = f + g$ and $k = fg$. Prove that

$$h(r) = f(r) + g(r) \quad \text{and} \quad k(r) = f(r)g(r).$$

[**Side note**: That is, you are being asked to prove that the function $\varphi_r : R[X] \to R$ given by $\varphi_r(f) = f(r)$ is a homomorphism. This function φ_r is called an **evaluation homomorphism**. Note that the point is that $(fg)(r)$ means to multiply the *polynomials* f and g according to $\cdot$ in $R[X]$, and then evaluate the result at r. On the other hand, $f(r)g(r)$ means to evaluate each of f and g at r and then multiply $f(r), g(r) \in R$ as a product in R. And similarly for $+$ in place of $\cdot$.]

Problem 4. (20 points) Saracino, Section 19, Problem 19.17
Let F be a field. For $f(X) = a_0 + a_1X + \cdots + a_nX^n \in F[X]$, define the *formal derivative* $f'(X)$ by

$$f'(X) = a_1 + 2a_2X + 3a_3X^2 + \cdots + na_nX^{n-1}.$$

(a) For $f, g \in F[X]$, define $h = f + g$. Prove that $h'(X) = f'(X) + g'(X)$

(b) For $f, g \in F[X]$, define $k = fg$. Prove that $k'(X) = f(X)g'(X) + f'(X)g(X)$

(c) Let $n \geq 1$ be a positive integer. Prove that the formal derivative of $[f(X)]^n$ is $n[f(X)]^{n-1} \cdot f'(X)$

[**Note**: For part (c), please do induction on $n \geq 1$, using part (b).]

(Optional Challenges and Office Hour Information on next page)

Optional Challenges (do NOT hand in): Saracino Problems 19.2(f), 19.7, 19.15, 19.18

Questions? You can ask in:

Class:
Section 01: MWF 9:00–9:50am, SMUD 014
Section 02: MWF 11:00–11:50am, SMUD 205

My office hours: in my office (SMUD 406):
Tue 1:30–3:00pm
Wed 1:30–3:00pm
Fri 1:30–2:30pm

Allison Tanguay's QCenter Drop-in Hours, in SMUD 208:
MWF 10am – noon
TuTh 1pm – 4pm

Math Fellow Drop-in Hours, in SMUD 208:

Sun	6:00–7:30pm	(Kevin)
Mon	7:30–9:00pm	(Claire)
Tue	8:30–10:00pm	(Aidan)
Wed	7:30–9:00pm	(Claire)
Thu	8:30–10:00pm	(Aidan)
Fri	6:00–7:30pm	(Kevin)

Also, you may email me any time at `rlbenedetto@amherst.edu`