Homework #19

Due **Tuesday, April 29** in Gradescope by **11:59 pm ET**

READ Sections 18–19 in Saracino

WATCH 1. Required: Video 37: Polynomial Terminology (20:42)

2. Required: Video 38: Proving Irreducibility (16:38)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (24 points) Saracino, Section 18, Problem 18.6
Prove that the ring $R = \{a + b\sqrt{2} \,|\, a, b \in \mathbb{Z}\}$ has precisely two automorphisms.

That is, find two different isomorphism $f_1, f_2 : R \to R$ (and prove they are both isomorphisms), and then prove that any arbitrary isomorphism $\varphi : R \to R$ must equal either f_1 or f_2.

[Note from RLB: you may take my word for it that R *is* a ring. Also, by Theorem 18.2(i), any automorphism $\varphi : R \to R$ must have $\varphi(1) = 1$. The real question is: what are the options for what $\varphi(\sqrt{2})$ could be?]

Problem 2. (6 points) Saracino, Section 18, Problem 18.15
Let $\varphi : R \to S$ be a (ring) homomorphism. Prove that φ is one-to-one if and only if $\ker \varphi = \{0_R\}$.

[Note from RLB: You may want to simply prove this if-and-only-if statement directly; the proof is not very long. But you may also be able to find an even quicker proof by quoting the corresponding fact about group homomorphisms. If you do it that second way, be sure you are clear about why all the claims you make along the way are legitimate.]

Problem 3. (14 points) Saracino, Section 18, Problem 18.22(a)
Let $\varphi : R \to S$ be a (ring) homomorphism, and let J be a prime ideal of S. If $\varphi^{-1}(J) \neq R$, prove that $\varphi^{-1}(J)$ is a prime ideal of R.

[Note from RLB: Yes, Theorem 18.4(iii) gives the "ideal" part of this, but since that theorem has no proof provided, please prove *both* that $\varphi^{-1}(J)$ is an ideal of R, *and* that it is prime.]

Problem 4. (14 points) Saracino, Section 18, Problem 18.28
In the proof of Theorem 18.10, we had an integral domain D and a set F called the field of fractions of D. Prove that the operations $+$ and $\cdot$ defined on F in that proof are well-defined.

Problem 5. (12 points) Saracino, Section 19, Problem 19.1
Let $f(X) = a_0 + a_1X + \cdots + a_rX^r \in \mathbb{Z}[X]$. Suppose $m/n \in \mathbb{Q}$, with $(m, n) = 1$. If m/n is a root of f, prove that $m|a_0$ and $n|a_r$.

Optional Challenges (do NOT hand in): Saracino Problems 17.34, 18.24, 18.29

(Office Hour Information on next page)

Questions? You can ask in:

Class:
Section 01: MWF 9:00–9:50am, SMUD 014
Section 02: MWF 11:00–11:50am, SMUD 205

My office hours: in my office (SMUD 406):
Tue 1:30–3:00pm
Wed 1:30–3:00pm
Fri 1:30–2:30pm

Allison Tanguay's QCenter Drop-in Hours, in SMUD 208:
MWF 10am – noon
TuTh 1pm – 4pm

Math Fellow Drop-in Hours, in SMUD 208:

Sun	6:00–7:30pm	(Kevin)
Mon	7:30–9:00pm	(Claire)
Tue	8:30–10:00pm	(Aidan)
Wed	7:30–9:00pm	(Claire)
Thu	8:30–10:00pm	(Aidan)
Fri	6:00–7:30pm	(Kevin)

Also, you may email me any time at `rlbenedetto@amherst.edu`