Homework #18

Due **Friday, April 25** in Gradescope by **11:59 pm ET**

READ Sections 17–18 in Saracino and the (optional!) **Theorem on Norms** handout

WATCH 1. Required: Video 34: Maximal Ideals and Fields (13:41)

2. Optional: Video 35: The Minimal Subfield (17:29)

3. Optional: Video 36: The First Isomorphism Theorem for Rings (7:20)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (17 points) Saracino, Section 17, Problem 17.9:
Let $R = \{q \in \mathbb{Q} \,|\, q = a/b \text{ with } a, b \in \mathbb{Z} \text{ and } b \text{ is odd}\}$. [You may take my word for it that R is a commutative ring with unity.] Prove that R has a unique maximal ideal.
[Hint from RLB: it's a principal ideal.]

Problem 2. (8 points) Saracino, Section 17, Problem 17.13:
Let I be an ideal of a ring R. Prove that the distributive laws hold in R/I.

Problem 3. (8 points) Saracino, Section 17, Problem 17.14:
Let R be a ring and I an ideal of R.

(a) If R is commutative, prove that R/I is commutative.

(b) If R has unity, prove that R/I has unity.

Problem 4. (14 points) Saracino, Section 17, Problem 17.22(a,b):
Let R be a commutative ring and X a subset of R. The **annihilator** of X is

$$\mathrm{Ann}(X) = \{r \in R \,|\, rx = 0 \text{ for every } x \in X\}.$$

(a) Prove that $\mathrm{Ann}(X)$ is an ideal of R.

(b) Let $R = \mathbb{Z}/12\mathbb{Z}$. Find $\mathrm{Ann}(\{2\})$.

[Note from RLB: on part (b), by 2, of course, Saracino means the element $12\mathbb{Z} + 2$ of the ring $\mathbb{Z}/12\mathbb{Z}$. This is because it is quite common to denote the unity element of a ring by 1, in which case the element $1 + 1$ is similarly named 2, as Saracino and I have called it here.]

Problem 5. (14 points) Saracino, Section 17, Problem 17.33:
Let R be a ring, and let I and J be ideals of R. Define $I + J = \{x + y \,|\, x \in I,\, y \in J\}$.

(a) Prove that $I + J$ is an ideal of R.

(b) Let $R = \mathbb{Z}$. Find $6\mathbb{Z} + 14\mathbb{Z}$. [And prove your answer, of course.]

Problem 6. (14 points) Saracino, Section 18, Problem 18.1(b,c,e):
Which of the following are ring homomorphisms? [Prove your answers, of course]

(b) $\varphi : \mathbb{C} \to \mathbb{C}$ by $\varphi(a + bi) = a - bi$

(c) $\varphi : \mathbb{C} \to \mathbb{R}$ by $\varphi(a + bi) = a$

(e) Let R be the ring of polynomials with real coefficients, and let $\varphi : R \to R$ by $\varphi(p(x)) = p'(x)$, the derivative of $p(x)$.

(Optional Challenges and Office Hour Information on next page)

Optional Challenges (do NOT hand in): Saracino Problems 17.27, 17.28(a), 17.30

Questions? You can ask in:

Class:
Section 01: MWF 9:00–9:50am, SMUD 014
Section 02: MWF 11:00–11:50am, SMUD 205

My office hours: in my office (SMUD 406):
Tue 1:30–3:00pm
Wed 1:30–3:00pm
Fri 1:30–2:30pm

Allison Tanguay's QCenter Drop-in Hours, in SMUD 208:
MWF 10am – noon
TuTh 1pm – 4pm

Math Fellow Drop-in Hours, in SMUD 208:

Sun	6:00–7:30pm	(Kevin)
Mon	7:30–9:00pm	(Claire)
Tue	8:30–10:00pm	(Aidan)
Wed	7:30–9:00pm	(Claire)
Thu	8:30–10:00pm	(Aidan)
Fri	6:00–7:30pm	(Kevin)

Also, you may email me any time at `rlbenedetto@amherst.edu`