Homework #16

Due **Tuesday, April 15** in Gradescope by **11:59 pm ET**

READ Sections 13, 16 in Saracino and the (optional!) **Other Group Topics** handout

WATCH 1. Optional: Video 29: Multiplication Modulo n: Extra (24:48)

2. Required: Video 30: Simple Groups (30:06)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (7 points) Saracino, Section 13, Problem 13.14:
Let G, H be finite groups, and let $\varphi : G \to H$ be an onto homomorphism.
Prove that $|H|$ divides $|G|$.

Problem 2. (9 points) Saracino, Section 13, Problem 13.19:
Let $\varphi : G \to K$ be a homomorphism. Prove that φ is one-to-one if and only if $\ker(\varphi) = \{e_G\}$.

Problem 3. (14 points) Saracino, Section 13, Problem 13.5:
Let G be the group of all real-valued functions on the real line, under addition of functions. Let $H = \{f \in G \mid f(0) = 0\}$.

(a) Prove that $H \lhd G$. (b) Prove that $G/H \cong \mathbb{R}$.

[Suggestion from RLB: do both parts in one fell swoop by defining a function $\varphi : G \to \mathbb{R}$ that you then prove is a homomorphism, is onto, and has kernel H.]

Problem 4. (11 points) (A useful fact for the next problem):
Let $G = \mathbb{Q}^\times$, and let $H = \{a/b \mid a, b \text{ are odd integers}\}$. Prove that for every $x \in G$, there exist unique numbers $k \in \mathbb{Z}$ and $h \in H$ such that $x = 2^k h$.
[Note: don't forget to prove **both** existence and uniqueness of k and h.]

Problem 5. (16 points) Saracino, Section 13, Problem 13.8:
Let $G = \mathbb{Q}^\times$, and let $H = \{a/b \mid a, b \text{ are odd integers}\}$, as in the previous problem.
[You may take my word for it that H is a subgroup of G.]
Prove that $G/H \cong \mathbb{Z}$.

[Suggestion from RLB: Use the previous problem, and define $\varphi : \mathbb{Q}^\times \to \mathbb{Z}$ by $\varphi(2^k h) = k$. Now prove φ is a well-defined homomorphism that is onto, and that $\ker \varphi = H$; then apply the Fundamental Theorem.]

Problem 6. (5 points) Saracino, Section 16, Problem 16.1:
Let R be a ring with unity 1_R. Prove that $(-1_R)a = -a$ for all $a \in R$.

(Problems continue on next page)

Problem 7. (8 points) Saracino, Section 16, Problem 16.13:
Let R be a ring with unity.

(a) Prove that the multiplicative identity element 1_R of R is unique.

(b) Let $a \in R^\times$ be a unit. Prove that the multiplicative inverse a^{-1} of a is unique.

Optional Challenges (do NOT hand in): Saracino Problems 13.4, 13.13, 13.20, 13.27

Questions? You can ask in:

Class:
Section 01: MWF 9:00–9:50am, SMUD 014
Section 02: MWF 11:00–11:50am, SMUD 205

My office hours: in my office (SMUD 406):
Tue 1:30–3:00pm
Wed 1:30–3:00pm
Fri 1:30–2:30pm

Allison Tanguay's QCenter Drop-in Hours, in SMUD 208:
MWF 10am – noon
TuTh 1pm – 4pm

Math Fellow Drop-in Hours, in SMUD 208:

Sun	6:00–7:30pm	(Kevin)
Mon	7:30–9:00pm	(Claire)
Tue	8:30–10:00pm	(Aidan)
Wed	7:30–9:00pm	(Claire)
Thu	8:30–10:00pm	(Aidan)
Fri	6:00–7:30pm	(Kevin)

Also, you may email me any time at `rlbenedetto@amherst.edu`