Homework #12
Due **Tuesday, March 25** in Gradescope by **11:59 pm ET**

READ $\boxed{\text{Sections 10–11 in Saracino}}$ and the $\boxed{\textbf{Groups of Order Six} \text{ handout}}$

WATCH 1. $\boxed{\text{Required: Video 23: The Class Equation}}$ (31:31)

2. $\boxed{\text{Optional: Video 24: Conjugacy Classes in } S_n}$ (32:49)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (8 points) Saracino, Section 10, Problem 10.15:
Let G be a finite group, and let H be a subgroup of G. Let K be a subgroup of H. Prove that $[G : K] = [G : H][H : K]$.

Problem 2. (8 points) Saracino, Section 10, Problem 10.23(a):
Let $S =$ the set of even integers, and let $T =$ the set of odd integers.
Prove that S and T have the same cardinality.
[Note from RLB: that is, define a function $f : S \to T$, and prove that your function is one-to-one and onto.]

Problem 3. (12 points) Saracino, Section 10, Problem 10.25:
Find the conjugacy classes in Q_8, and write down the class equation for Q_8.

Problem 4. (14 points) Saracino, Section 10, Problem 10.28:
Let p be a prime number, and let n be a positive integer. Let G be a group with $|G| = p^n$. Use the class equation to prove that $|Z(G)|$ is divisible by p.

Problem 5. (12 points) Saracino, Section 10, Problem 10.29:
Let p be a prime number and let G be a group such that $|G| = p^2$. Prove that G is abelian.
[Suggestion from RLB: Use the previous problem. Then, using Lagrange's Theorem, for any $a \in G$ with $a \notin Z(G)$, what can you say about the centralizer $Z(a)$?]

Problem 6. (8 points) Saracino, Section 11, Problem 11.1:
Recall that $SL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}) \,\middle|\, ad - bc = 1 \right\}$.
We have already seen that $SL(2, \mathbb{R})$ is a subgroup of $GL(2, \mathbb{R})$. Prove that $SL(2, \mathbb{R}) \triangleleft GL(2, \mathbb{R})$.

Problem 7. (8 points) Saracino, Section 11, Problem 11.2:
Let H be the subgroup of $G = GL(2, \mathbb{R})$ consisting of all matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ such that $ad \neq 0$. Is H a normal subgroup of G? Why or why not?

(Optional Challenges and Office Hour Information on next page)

Optional Challenges (do NOT hand in): Saracino Problems 10.32, 10.33

Questions? You can ask in:

Class:
  Section 01: MWF 9:00–9:50am, SMUD 014
  Section 02: MWF 11:00–11:50am, SMUD 205

My office hours: in my office (SMUD 406):
  Tue 1:30–3:00pm
  Wed 1:30–3:00pm
  Fri 1:30–2:30pm

Allison Tanguay's QCenter Drop-in Hours, in SMUD 208:
  MWF 10am – noon
  TuTh 1pm – 4pm

Math Fellow Drop-in Hours, in SMUD 208:
  Sun 6:00–7:30pm   (Kevin)
  Mon 7:30–9:00pm   (Claire)
  Tue 8:30–10:00pm   (Aidan)
  Wed 7:30–9:00pm   (Claire)
  Thu 8:30–10:00pm   (Aidan)
  Fri   6:00–7:30pm   (Kevin)

Also, you may email me any time at `rlbenedetto@amherst.edu`