Homework #1

Due **Friday, January 31** in Gradescope by **11:59 pm ET**

READ Sections 0–2 in Saracino

WATCH 1. Video 1: A Sample Proof (11:42)

2. Video 2: Induction: An Example (12:22)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (3 points) Saracino, Section 0, Problem 0.1:
Let $S = \{2, 5, \sqrt{2}, 25, \pi, 5/2\}$ and $T = \{4, 25, \sqrt{2}, 6, 3/2\}$. Find $S \cap T$ and $S \cup T$.

Problem 2. (7 points) Saracino, Section 0, Problem 0.5:
Let A, B, C be sets. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Problem 3. (9 points) Saracino, Section 0, Problem 0.8:
Prove that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$ for all $n \geq 1$.

Problem 4. (12 points) Saracino, Section 1, Problem 1.3(a,c,g):
Which of the following are binary operations on the given set? In each case, if $*$ **is** a binary operation, give a short explanation why, but not a formal proof. If $*$ is **not** a binary operation, give one explicit example of $x * y$ that shows this failure.

(a): $S = \mathbb{Z}$ and $a * b = a + b^2$.

(c): $S = \mathbb{R}$ and $a * b = \dfrac{a}{a^2 + b^2}$.

(g): $S = \{1, -2, 3, 2, -4\}$ and $a * b = |b|$.

Problem 5. (20 points) Saracino, Section 2, Problem 2.1(b,c,f,g):
Which of the following are groups? For each part, if you say it **is** a group, give a short explanation, but not a formal proof, of why each of the four group axioms holds (binary operation, associative, identity, inverses), including saying what the identity element is and what the inverse of an arbitrary element g is. If you say it is **not** a group, give one explicit example of one axiom failing.

(b): The set $3\mathbb{Z}$ of integers that are multiples of 3, under addition.

(c): $\mathbb{R} \smallsetminus \{0\}$ under the operation $a * b = |ab|$.

(f): The set of all pairs (x, y) of real numbers, under the operation $(x, y) * (z, w) = (x + z, y - w)$.

(g): The set of all pairs (x, y) of real numbers such that $y \neq 0$, under the operation
$(x, y) * (z, w) = (x + z, yw)$.

Problem 6. (14 points) Saracino, Section 2, Problem 2.1(i):
The set is $\mathbb{Z}$, and the operation is $a * b = a + b - 1$. Perhaps surprisingly, this one **is** a group. Give a formal proof of this fact.

Optional Challenges (do NOT hand in): Saracino Problems 0.11, 0.18, 2.1(e,h)

Questions? You can ask in:

Class:
Section 01: MWF 9:00–9:50am, SMUD 014
Section 02: MWF 11:00–11:50am, SMUD 205

My office hours: in my office (SMUD 406):
Tue 1:30–3:00pm
Wed 1:30–3:00pm
Fri 1:30–2:30pm

(Math Fellow and QCenter office hours will start soon, too!)

Also, you may email me any time at `rlbenedetto@amherst.edu`