Math 350, Sections 01,03, Fall 2021

Final Exam
Due THURSDAY, DECEMBER 16, 2021, at 11:59PM ET on gradescope
Take-home: open book and open notes

Instructions: Do all **eight numbered problems** (totalling 200 points), and in addition, as a ninth "question," **write out and sign the academic honesty pledge found later in this document**. There are also two optional bonus problem worth a total of 4 points.

Write your answers neatly and legibly, and tag them correctly on gradescope.
You must fully justify your answers. Simple algebraic deductions need not be explained, but other leaps of logic require a word or two of justification. You may use theorems (and lemmas, etc.) from class or the book to do so; however, make sure you verify all the relevant hypotheses of any theorem you use, and reference the source. (E.g., "By Theorem 12.1," or "By Lagrange's Theorem," or "By Exercise 13.22," etc.) Unless otherwise noted, **you may quote only theorems that we covered or exercises that were assigned (not challenge problems or results from other sections)**. If you are not sure whether some argument or statement requires further justification, please ask me about it.
In working on the problems, you **may** use:

- the book (Saracino, Sections 0–13 and 16–20),
- handouts and videos from the course, and
- **your own notes**.

But as a matter of Academic Honesty, you may NOT use

- other books,
- online information besides this course's webpages, or
- any other outside sources,

and you **may NOT discuss the problems** with anyone other than me. (Not even to mention casually that, say, "Number 14 is pretty easy.")

However,

- You should feel free to talk to me about any problem. I will be much less helpful than I am for homework assignments, but I will be happy to clarify things. You may ask me anything you want; I will decide how much I can answer.
- You may talk to Allison or the Fellows about concepts from the course, old homework problems, and other old course materials, as long as you are careful to stay away from the actual exam problems. If I connected you with a peer tutor, the same applies to them.

The exam is due at **11:59 PM ET** on the Thursday of exam period, on gradescope. You may submit it as early as you want, even days early, but **no extensions will be granted**. Any exam not submitted on time will be graded as a zero.

I strongly recommend that you

consider the deadline to be 8pm ET, Thursday, Dec 16

so that you have a four-hour grace period if you miss that deadline.

1. **(20 points)** For each of the following groups G, decide whether or not G has an element of order 6. If so, give an example of such an element, and prove that it indeed has order 6. If not, prove that there is no such element.

(a) C_{98} (Recall C_n is what the book calls $\mathbb{Z}_n$)

(b) $C_3 \times C_{15}$

(c) $C_9 \times Q_8$ (Recall Q_8 is the group of quaternion units.)

(d) S_5 (Recall S_n is the permutation group on n symbols.)

2. **(10 points)** Let G be a nonabelian group of order 27, and let $a \in G$. Prove that $a^9 = e$.

3. **(20 points)** Let G and H be groups, and let $\varphi : G \to H$ be a homomorphism. Recall that $Z(G)$ denotes the center of G, and $Z(H)$ denotes the center of H.

(a) If φ is injective, prove that $\varphi^{-1}(Z(H)) \subseteq Z(G)$.

(b) If φ is sujective, prove that $\varphi(Z(G)) \subseteq Z(H)$.

4. **(30 points)** Let G be an abelian group, and define

$$K = \{x \in G \,|\, x^{2^n} = e \text{ for some integer } n \geq 0\}.$$

Equivalently, an element $x \in G$ belongs to K if and only $o(x)$ is a power of 2.

(a) Prove that K is a subgroup of G.

(b) Prove that no element of G/K has even order.

(*Suggestion*: For part(b), given an element $Ky \in G/K$ and an integer $m \geq 1$ such that $(Ky)^{2m}$ is the identity of G/K, prove that $(Ky)^m$ is already the identity. That won't solve everything, but it should be a helpful step.)

5. **(30 points)** Let R be a commutative ring with unity, and let $I, J \subseteq R$ be ideals. The **product ideal** IJ is defined to be

$$IJ = \{x_1y_1 + \cdots + x_ny_n \,|\, n \geq 1, \; x_i \in I, \text{ and } y_i \in J\}.$$

(That is, IJ consists of all finite sums of products xy for $x \in I$ and $y \in J$.)

(a) Prove that IJ **is** an ideal of R.

(b) Prove that $IJ \subseteq I \cap J$.

(c) If $I + J = R$, prove that $IJ = I \cap J$.
(Recall that $I + J = \{x + y \,|\, x \in I, y \in J\}$, as defined in Exercise 17.33.)

6. **(25 points)** Let R be a commutative ring with unity, and let $I \subseteq R$ be an ideal. Prove that **the following two statements are equivalent**:

(i) $R^\times = R \smallsetminus I$. (That is, the set of units of R is precisely the complement of I.)

(ii) I is a maximal ideal of R, and every proper ideal of R is contained in I.

7. **(35 points)** Let R be a commutative ring, and let $I \subseteq R$ be an ideal. Define

$$J = \{r \in R \,|\, \text{there is an integer } n \geq 1 \text{ such that } r^n \in I\}.$$

(a) Prove that J is an ideal of R.

(b) Prove that the quotient ring R/J contains no nonzero nilpotent elements.

(*Suggestion*: For part (a), it may be helpful to state and prove a lemma: that for any $x, y \in R$ and any $k \geq 1$, there are integers $c_0, \ldots, c_k \in \mathbb{Z}$ such that $(x+y)^k = \sum_{i=0}^{k} c_i x^i y^{k-i}$.)

8. **(30 points)** Recall that $\mathbb{F}_2 = \{0, 1\}$ denotes the field of two elements. List all polynomials in $\mathbb{F}_2[X]$ of degree 4, and determine which of them are irreducible.
Make sure your list is complete; and as always, you must justify your answer for each polynomial.

9. (**Absolutely required**): Write out the following statement and sign your name to it:

I have read, understood, and followed the Academic Honesty instructions on the cover page of this Final Exam.

BONUS A. **(2 points)** Recall that A_6 denotes the alternating group on 6 objects, and S_3 is the symmetric group on 3 objects. Find an **injective** homomorphism $\varphi : S_3 \to A_6$.

BONUS B. **(2 points)** Let $R = \{a + b\sqrt{7} \,|\, a, b \in \mathbb{Z}\}$, viewed as a subset of $\mathbb{R}$, with addition and multiplication as in $\mathbb{R}$. You may take my word for it that R is a commutative ring with unity. Prove that R has infinitely many units.