Solutions to Midterm Exam 2

1. Let $\sigma = (1,3,9)(2,7,6) \in S_9$ and $\tau = (1,9,4,7)(2,3)(6,8) \in S_9$.
   (a) Write $\sigma\tau$ as a product of **disjoint** cycles.
   (b) Compute the **order** of each of σ, τ, and $\sigma\tau$.
   (c) Decide whether each of σ, τ, and $\sigma\tau$ is an **even** or **odd** permutation.
   (d) Find an element of S_9 of order 15.

Solution. (a): $\sigma\tau = (1,3,9)(2,7,6)(1,9,4,7)(2,3)(6,8) = \boxed{(2,9,4,6,8)(3,7)}$

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(b): $o(\sigma) = \text{lcm}(3,3) = \boxed{3}$
$o(\tau) = \text{lcm}(4,2,2) = \boxed{4}$
$o(\sigma\tau) = \text{lcm}(5,2) = \boxed{10}$

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(c): 3-cycles are even, so σ is even + even = $\boxed{\text{even}}$
Similarly, τ is odd + odd + odd = $\boxed{\text{odd}}$
And $\sigma\tau$ is even + odd = $\boxed{\text{odd}}$

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(d): The order of $\boxed{(1,2,3)(4,5,6,7,8) \in S_9}$ is $\text{lcm}(3,5) = 15$

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2. Let G be a group, and let $H, K \lhd G$ be normal subgroups of G. Define
$$HK = \{hk : h \in H \text{ and } k \in K\}.$$
Prove that HK is a normal subgroup of G. (I.e. prove *both* subgroup *and* normal.)

Proof. **(Nonempty)**: We have $e \in H$ and $e \in K$, so $ee \in HK$.

(Closed): Given $x, y \in HK$, there exist $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $x = h_1k_1$ and $y = h_2k_2$.
Since $H \lhd G$, there exists $h_3 \in H$ such that $k_1h_2 = h_3k_1$.
Thus, $xy = h_1k_1h_2k_2 = h_1h_3k_1k_2 \in HK$, since $h_1h_3 \in H$ and $k_1k_2 \in K$.

(Inverses): Given $x \in HK$, there exist $h \in H$ and $k \in K$ such that $x = hk$.
Since $H \lhd G$, there exists $h' \in H$ such that $k^{-1}h^{-1} = h'k^{-1}$.
Thus, $x^{-1} = (hk)^{-1} = k^{-1}h^{-1} = h'k^{-1} \in HK$.

(Normal): Given $x \in HK$ and $g \in G$, there exist $h \in H$ and $k \in K$ such that $x = hk$.
Then $gxg^{-1} = ghkg^{-1} = (ghg^{-1})(gkg^{-1}) \in HK$, since $ghg^{-1} \in H$ and $gkg^{-1} \in K$. QED

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3. Let G and H be groups, and let $\varphi : G \to H$ and $\psi : G \to H$ be homomorphisms. Define
$$K = \{x \in G \,|\, \varphi(x) = \psi(x)\}.$$
Prove that K is a subgroup of G.

Proof. **(Nonempty)**: We have $e_G \in K$, since $\varphi(e_G) = e_H = \psi(e_G)$.

(Closed): Given $x, y \in K$, then $\varphi(xy) = \varphi(x)\varphi(y) = \psi(x)\psi(y) = \psi(xy)$, so $xy \in K$.

(Inverses): Given $x \in K$, then $\varphi(x^{-1}) = \big(\varphi(x)\big)^{-1} = \big(\psi(x)\big)^{-1} = \psi(x^{-1})$,
so $x^{-1} \in K$. QED

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4. Let $G = D_6 = \{e, f, f^2, f^3, f^4, f^5, g, fg, f^2g, f^3g, f^4g, f^5g\}$, the 12-element dihedral group of rotations and flips of a regular hexagon. Let $H = \langle f^3 \rangle = \{e, f^3\}$.
It is a fact, which you may assume, that $H \lhd G$.

4a. List all of the elements of the quotient group G/H.
Make sure your answer makes clear how many **distinct** elements there are.

4b. Let $x = Hf$, which is an element of G/H. Compute $o(x)$.

4c. Let $y = Hfg$, which is an element of G/H. Compute $o(y)$.

Solution. (a): G/H must have $[G : H] = |G|/|H| = 12/2 = 6$ elements. They are:

$$\boxed{He = Hf^3, \quad Hf = Hf^4, \quad Hf^2 = Hf^5, Hg = Hf^3g, \quad Hfg = Hf^4g, \quad Hf^2g = Hf^5g}$$

All 12 elements of G appear above, and each coset is expressed both as Ha and Hf^3a, so the list is complete.
[Note: there are other ways to show the list is complete, with no repeats, without writing each coset in two different ways.]

(b): We have $x = Hf \neq He$, since $fe^{-1} = f \notin H$.
We have $x^2 = Hf^2 \neq He$, since $f^2e^{-1} = f^2 \notin H$.
We have $x^3 = Hf^3 = He$, since $f^3e^{-1} = f^3 \in H$.
Thus, $\boxed{o(x) = 3}$

(c): We have $y = Hfg \neq He$, since $fge^{-1} = fg \notin H$.
We have $y^2 = H(fg)^2 = He$, since $(fg)^2 = f(gf)g = f(f^{-1}g)g = g^2 = e$.
Thus, $\boxed{o(y) = 2}$

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5. Let G be a group, and let $H \lhd G$ be a normal subgroup. Suppose that the quotient group G/H is abelian. For any $a, b \in G$, prove that $aba^{-1}b^{-1} \in H$.

Proof. Given $a, b \in G$, we have

$$Hab = HaHb = HbHa = Hba,$$

where the second equality is because G/H is abelian, and the first and last are by definition of the group law on G/H.
Thus, by the coset relation, we have $(ab)(ba)^{-1} \in H$, i.e., $aba^{-1}b^{-1} \in H$. QED

OPTIONAL BONUS. Let G be a group of order 30, and suppose that the center $Z(G)$ has order 5. Let $a \in G$ such that $a \notin Z(G)$. Prove that the centralizer $Z(a)$ is cyclic.

Proof. Let $n = |Z(a)|$. Since $Z(a)$ is a subgroup of G, we have $n|30$ by Lagrange.

The subgroup $Z(a)$ contains the subgroup $Z(G)$, since every $z \in Z(G)$ commutes with every element of G, and so in particular $za = az$, so that $z \in Z(a)$. Thus, we must also have $5|n$ by Lagrange.

Therefore, n is one of 5, 10, 15, 30.

However, $a \in Z(a)$, since a commutes with itself (i.e., since $aa = aa$), and since $a \notin Z(G)$, it follows that $Z(G) \subsetneq Z(a)$, and hence $n > 5$. That is, n is one of 10, 15, 30.

If $n = 30$, then $Z(a) = G$, meaning that every $g \in G$ satisfies $ag = ga$. But then $a \in Z(G)$, a contradiction.

So we must have either $n = 10$ or $n = 15$. That is, $n = 5p$ where p is either 2 or 3.

Another Theorem says that $Z(G)$ is a normal subgroup of G, and hence a normal subgroup of $Z(a)$. Note that $[Z(a) : Z(G)] = |Z(a)|/|Z(G)| = p$, which is prime (since $p = 2$ or $p = 3$). Note also that the coset $Z(G)a$ is a non-identity element of the quotient group $Z(a)/Z(G)$, since $Z(G)a \neq Z(G)e$, as $ae^{-1} = a \notin Z(G)$. Thus, by a corollary of Lagrange applied to the (prime order) quotient group $Z(a)/Z(G)$, we have that $Z(G)a$ is a generator of $Z(a)/Z(G)$. In particular, $o(Z(G)a) = p$, and so $Z(G)a^p = (Z(G)a)^p = Z(G)e$, which gives us $a^p = a^p e^{-1} \in Z(G)$.

Case 1. If $a^p \neq e$, then a^p is a nonidentity element of $Z(G)$, so by the same corollary of Lagrange again, we have $\langle a^p \rangle = Z(G)$, and hence $o(a^p) = 5$. Therefore, $a^{5p} = e$, so that $o(a)|5p$ [which we already knew from Lagrange, because $|Z(a)| = 5p$] and hence $o(a)$ is one of 1, 5, p, or $5p$. However, since $a^p \neq e$, we have $o(a) \neq 1, p$, so either $o(a) = 5$ or $o(a) = 5p$. On the other hand, since $o(Z(G)a) = p \nmid 5$ (in the quotient group), we have $Z(G)a^5 \neq Z(G)e$, so that $a^5 \notin Z(G)$, and in particular, $o(a) \neq 5$. Therefore, $o(a) = 5p = |Z(a)|$. Thus, a is a generator for $Z(a)$, proving that $Z(a)$ is cyclic.

Case 2. Otherwise, we have $a^p = e$. Pick $b \in Z(G) \smallsetminus \{e\}$. Since $|Z(G)| = 5$ is prime, a corollary of Lagrange says that $Z(G) = \langle b \rangle$. Let $c = ba \in Z(a)$.
Note that $Z(G)c = Z(G)a$, since $ca^{-1} = b \in Z(G)$. Thus, $o(Z(G)c) = o(Z(G)a) = p$. As in Case 1, it follows that $c^5 \notin Z(G)$, and hence $o(c) \neq 1, 5$. In addition, $c^p = (ba)^p = b^p a^p = b^p e = b^p \neq e$, where the second equality is because $b \in Z(G)$ (and hence $ab = ba$), and the inequality is because $o(b) = 5$, and $5 \nmid p$. Therefore, we also have $o(c) \neq p$.
But since $c \in Z(a)$, we have $o(c)|5p$. Because $c \neq 1, 5, p$, it follows that $o(c) = 5p$. As in Case 1, it follows that c is a generator for $Z(a)$, proving that $Z(a)$ is cyclic. QED