Math 350, Section 02, Fall 2022

**Solutions to Exam 2**

1. **(20 points)** Let $f = (3,7,4)(2,9,6,5) \in S_9$ and $g = (1,9,3,8,5) \in S_9$.

   1a. Write $fg$ as a product of disjoint cycles.

   1b. Find the orders of each of $f$, $g$, and $fg$.

   1c. Suppose that $h \in S_9$ is a cycle of order 9. Is the permutation $ghf$ even or odd?

**Solutions**. (a): $fg = (3,7,4)(2,9,6,5)(1,9,3,8,5) = \boxed{(1,6,5)(2,9,7,4,3,8)}$

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(b): $o(f) = \mathrm{lcm}(3,4) = \boxed{12}$ $\quad o(g) = \boxed{5}$ $\quad$ and $o(fg) = \mathrm{lcm}(3,6) = \boxed{6}$

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(c): $ghf$ is the composition of:

a 5-cycle (which is even), a 9-cycle (even), a 3-cycle (even), and a 6-cycle (odd),

and hence $ghf$ is even+even+even+odd= $\boxed{\text{odd}}$

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2. **(20 points)** Let $G = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R} \right\} \subseteq GL(2,\mathbb{R})$. You may assume that $G$ is a subgroup of $GL(2,\mathbb{R})$. Define $\varphi : \mathbb{R} \to G$ by

$$\varphi(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}.$$

Prove that $\varphi$ is an isomorphism.

**Proof**. (**Homom.**): First, we observe that $\varphi$ actually **is** a function between the two sets. For any $a \in \mathbb{R}^\times$, clearly $\varphi(a)$ as above is a $2 \times 2$ matrix whose four (real) entries satisfy the conditions for belonging to $G$.
Now, given any $a, b \in \mathbb{R}$, we have

$$\varphi(a)\varphi(b) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} = \varphi(a+b),$$

and hence $\varphi$ is a homomorphism.

(**1-1**): Given $a, b \in \mathbb{R}$ with $\varphi(a) = \varphi(b)$, we have $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. Looking at the upper right entries, then, we have $a = b$. Thus, $\varphi$ is one-to-one.

(**Onto**): Given $A \in G$, we may write $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ for some $a \in \mathbb{R}$. Thus, $\varphi(a) = A$, and hence $\varphi$ is onto. QED

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3. **(20 points)** Let $G$ be a group, let $X \lhd G$ be a normal subgroup, and let $Y \subseteq G$ be a subgroup. Define

$$H = \{xy \,|\, x \in X \text{ and } y \in Y\}.$$

Prove that $H$ is a subgroup of $G$.

**Proof**. **(Nonempty)**: We have $e \in X$ and $e \in Y$, so $ee \in H$.

**(Closed)**: Given $h_1, h_2 \in H$, there exist $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ such that $h_1 = x_1y_1$ and $h_2 = x_2y_2$. Then

$$h_1h_2 = x_1y_1x_2y_2 = \big(x_1(y_1x_2y_1^{-1})\big)(y_1y_2) \in H$$

because we have $y_1 x_2 y_1^{-1} \in X$ since $X \lhd G$, and hence $x_1(y_1 x_2 y_1^{-1}) \in X$ as well; and because meanwhile $y_1 y_2 \in Y$.

**(Inverses)**: Given $h \in H$, there exist $x \in X$ and $y \in Y$ such that $h = xy$. Then

$$h^{-1} = y^{-1}x^{-1} = \big(y^{-1}x^{-1}y\big)y^{-1} \in H,$$

because we have $x^{-1} \in X$ and hence $y^{-1}x^{-1}y \in X$ since $X \lhd G$; and because meanwhile $y^{-1} \in Y$.

QED

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4. **(20 points)** Let $G = C_2 \times C_4$, and let $H$ be the cyclic subgroup $H = \langle (1,2) \rangle$.

4a. List all of the elements of $H$.

4b. List all of the elements of the quotient group $G/H$.

4c. Compute the order of the element $H + (1,1)$ of $G/H$.

**Solutions**. (a): Since $(1,2) + (1,2) = (0,0)$, we have $\boxed{H = \{(0,0),(1,2)\}}$

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(b): Since $|H| = 2$, we have that $G/H$ has $[G : H] = 8/2 = 4$ elements. The four distinct elements are

$$H + (0,0) = H + (1,2) = \{(0,(1,2)\}, \qquad H + (0,1) = H + (1,3) = \{(0,1),(1,3)\},$$
$$H + (0,2) = H + (1,0) = \{(0,2),(1,0)\}, \qquad H + (0,3) = H + (1,1) = \{(0,3),(1,1)\}.$$

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(c). $H + (0,0)$ is the identity. For $H + (1,1) \neq H(0,0)$, we compute

$$\begin{aligned} 2[H + (1,1)] &= H + (0,2) \neq H + (0,0) && \text{(because } (0,2) - (0,0) = (0,2) \notin H\text{)},\\ 3[H + (1,1)] &= H + (1,3) \neq H + (0,0) && \text{(because } (1,3) - (0,0) = (1,3) \notin H\text{)},\\ 4[H + (1,1)] &= H + (0,0), \end{aligned}$$

and therefore $\boxed{H + (1,1) \text{ has order } 4}$

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**Note**: Alternatively in (c), we could apply (a corollary of) Lagrange's Theorem to note that $o(H + (1,1))$ must divide $|G/H| = 4$, so $o(H + (1,1))$ is one of $1, 2, 4$.
Then, after computing $2[H + (1,1)] = H + (0,2) \neq H + (0,0)$, it follows that the order is not 1 or 2, and hence must be 4.

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5. **(20 points)** Let $G$ be a group, and let $H, K \subseteq G$ be subgroups. Suppose that $|H| = 30$ and $|K| = 33$.

5a. Prove that $|H \cap K|$ is either 1 or 3.

5b. Prove that $H \cap K$ is cyclic.

**Proof**. (a): We know that $H \cap K$ is a subgroup of $G$ and hence a group itself. Therefore, being a subset of both $H$ and $K$, it is also a subgroup of both $H$ and $K$.

Let $m = |H \cap K|$. By Lagrange's Theorem, we have $m \big| |H|$ and $m \big| |K|$. That is $m|30$ and $m|33$, so $m|\gcd(30,33)$, i.e., $m|3$. Since $m$ is positive and 3 is prime, it follows that $m = 1$ or $m = 3$. QED (a)

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(b): If $|H \cap K| = 1$, then $H \cap K = \{e\} = \langle e \rangle$, which is cyclic.
Otherwise, by part (a) we have $|H \cap K| = 3$. Since 3 is prime, by a corollary of Lagrange, it follows that $H \cap K$ is cyclie. QED (b)

**OPTIONAL BONUS. (2 points.)** Recall that $D_{15}$ denotes the dihedral group of order 30 (rotations and flips of a regular 15-gon), and $C_{15}$ denotes the cyclic group of order 15. Let $\varphi : D_{15} \to C_{15}$ be a homomorphism. Prove that $\varphi(x) = 0$ for all $x \in D_{15}$.

**Proof**. For each $i = 0, \ldots, 14$, consider the flip $h = f^i g \in D_{15}$. We have $o(h) = 2$, so $o(\varphi(h))|2$ [This is by a Theorem in Section 12, or alternatively from scratch by observing that $\varphi(h)^2 = \varphi(h^2) = \varphi(e) = 0$.]
But $2 \nmid 15 = |C_{15}|$, so $o(\varphi(h)) \neq 2$. Therefore $o(\varphi(h)) = 1$, meaning that $\varphi(h) = 0$.
In particular, both $\varphi(g)$ and $\varphi(fg)$ are 0. Therefore,

$$\varphi(f) = \varphi(fg^2) = \varphi(fg) + \varphi(g) = 0 + 0 = 0.$$

Hence, for every $x \in D_{15}$, since there are integers $i, j \in \mathbb{Z}$ such that $x = f^i g^j$, we have

$$\varphi(x) = \varphi(f^i g^j) = i\varphi(f) + j\varphi(g) = i(0) + j(0) = 0 + 0 = 0.$$

QED