## Rings: Basic Definitions

This handout is a quick reference sheet for basic terminology about rings.

---

**Definition.** A ring is a set $R$ together with two binary operations on $R$, denoted $+$ and $\cdot$, satisfying the following properties:

0. $+$ is indeed a binary operation: for all $x, y \in R$, we have $x + y \in R$.
1. $+$ is associative: for all $x, y, z \in R$, we have $(x + y) + z = x + (y + z)$.
2. $+$ has identity: there exists $0 \in R$ such that for all $x \in R$, we have $x + 0 = 0 + x = x$.
3. $+$ has inverses: for all $x \in R$, there exists $-x \in R$ such that $x + (-x) = (-x) + x = 0$.
4. $+$ is commutative: for all $x, y \in R$, we have $x + y = y + x$.
5. $\cdot$ is indeed a binary operation: for all $x, y \in R$, we have $x \cdot y \in R$.
6. $\cdot$ is associative: for all $x, y, z \in R$, we have $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
7. distributive laws: for all $x, y, z \in R$, we have $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

---

**Notes:**

- Officially, $(R, +, \cdot)$ is a ring, but we often abbreviate, saying simply that $R$ is a ring.
- Properties 0–4 can be summarized by saying that $(R, +)$ is an abelian group.
- The additive identity 0 is **always** called 0 (or $0_R$), and **never** called $e$.
- The additive inverse $-x$ is **always** called $-x$, and **never** called $x^{-1}$.
- As in high school algebra, we often write $x - y$ for $x + (-y)$.
- As in high school algebra, we often omit the symbol $\cdot$, but we **never** omit the symbol $+$.
- As in high school algbera, in the absence of parentheses, we do the $\cdot$ operation first. For example: $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

---

Notably missing from properties 0–7 above are any claims that the multiplication operation $\cdot$ has an identity, has inverses, or is commutative. We have special words for those scenarios:

**Definitions.** Let $R$ be a ring [implicitly, with operations $+$ and $\cdot$].

8. If $\cdot$ is commutative ($\forall x, y \in R$, we have $xy = yx$), we say $R$ is a commutative ring
9. If $\cdot$ has identity ($\exists 1 \in R$ s.t. $\forall x, y \in R$, we have $x1 = 1x = x$), we say $R$ is a ring with unity or, for short, a ring with 1
10. If $R$ is a ring with unity, with $1 \neq 0$, and if every **nonzero** element of $R$ has a multiplicative inverse ($\forall x \in R \setminus \{0\}$, $\exists y \in R$ s.t. $xy = yx = 1$), then we say $R$ is a division ring or a skew field
    For any ring with 1, we write $x^{-1}$ for the multiplicative inverse of $x$ (if it exists).
11. If $R$ is a commutative division ring (i.e., all of 0–10 hold), we say $R$ is a field

---

**Notes:**

- We **never** call a commutative ring abelian. "Abelian" is reserved for groups only.
- The multiplicative identity $1 \in R$ can also be denoted $1_R$, or perhaps something like $I$ (if elements of $R$ are matrices) or id (if elements of $R$ are functions), but it is usually **not called** $e$.
- In a ring $R$ with unity, if $x \in R$ has a multiplicative inverse $x^{-1} \in R$, we say $x$ is a unit
  The set of all units in $R$ forms a group, denoted $R^\times$. Its identity element is 1.
- Don't mix up the words **unity** (the multiplicative identity $1 \in R$, if it exists) and and **unit** (an element $x \in R$ having a multiplicative inverse).

When doing algebraic manipulations in rings, properties 0–7 say you can mostly proceed according to high school algebra rules, but you have to be careful if you don't have properties 8–10.

For example, you can't just replace $xy$ by $yx$ unless you know $R$ is commutative. You also can't just "divide by $x$"; instead, you first need to know that $x$ is a unit (i.e., invertible), and then you **multiply** by $x^{-1}$, specifically on the right or specifically on the left.

So for higher-level manipulations, you may need to adjust your intuitions a little bit. Fortunately, though, the following familiar fact (from class, and also Theorem 16.1(a) in Saracino) still holds for all rings:

**Proposition**. Let $R$ be a ring. Then for every $x \in R$, we have $0x = x0 = 0$

**Notes and Consequences:**

- This is why (optional) property 10 only asks for **nonzero** elements of $R$ to be units.
- If $1 = 0$ in $R$, then $R = \{0\}$, i.e., $R$ is the trivial ring
  This is why (optional) property 10 requires $1 \neq 0$.
- If 0 is a unit in $R$, then again $R$ has to be trivial.

---

As presented in some examples in the book and in class, it **can** happen in some rings $R$ that there are nonzero elements $x, y \in R$ such that $xy = 0$. This phenomenon deserves a name:

**Definition.** Let $R$ be a ring, and let $x \in R$.

- If there exists a **nonzero** $y \in R \setminus \{0\}$ such that either $xy = 0$ or $yx = 0$ (or both), then we say that $x$ is a zero-divisor
- If there exists a positive integer $n \geq 1$ such that $x^n = 0$, then we say that $x$ is nilpotent

[Of course, as usual, $x^n$ denotes $x \cdot x \cdot \cdots \cdot x$; for example, $x^3 = x \cdot x \cdot x$.]

---

One more basic definition:

**Definition.** Let $R$ be a **commutative ring with unity**,
and also suppose $1 \neq 0$ [i..e, suppose that $R$ is **not the trivial ring**].
Suppose further that $R$ **has no nonzero zero-divisors**.
Then we say $R$ is an integral domain or sometimes simply a domain

---

**Notes:**

- Every field is an integral domain. [Can you prove that?]
- Not every integral domain is a field. The archetypal example is $R = \mathbb{Z}$ (with usual $+, \cdot$).
- In fact, the term "**integral** domain" is meant to suggest the ring $\mathbb{Z}$ of **integers**.

In general, most elements of an integral domain $R$ do **not** have multiplicative inverses in $R$. (See, for example, the archetypal example $R = \mathbb{Z}$ of an integral domain.)

However, the fact that the only zero-divisor is 0 itself means that whenever you have an equation like $xy = 0$ in an integral domain, you can deduce that **either** $x = 0$ **or** $y = 0$ (or both), just by the domain property (and **not** by multiplying both sides by an inverse).