Math 350, Spring 2025 Professor Rob Benedetto

Review of Sets and Proofs

Expressions, Statements, and Implications.

- A mathematical *expression* is a mathematical noun, like x or 14 or $\sum_{n=1}^{\infty}(-1)^n n^{-3}$ or $\int_0^{3x}\cos(t^2)\,dt$ or $\{x\in\mathbb{Q} : 1<x<5\}$.
- a mathematical *statement* is an actual sentence with nouns (i.e., expressions) **and a verb** (like $=$ or $<$ or $\in$ or $\subseteq$).

For example, $x\in\mathbb{Q}$ is a statement; so is $x^2<1$; and so is $5=4$. Of course, $5=4$ is a **false** statement, but it is still a statement. By contrast, x^2+5 is **not** a statement, because it has no verb.

One special sort of mathematical verb is "*implies*," also denoted $\Rightarrow$. Whereas the verb $=$ goes between two *expressions*, the verb $\Rightarrow$ goes between two *statements*. So $(x+3)(x-1)\Rightarrow x^2+2x-3$ doesn't make sense; but it *does* make sense to say $x>3\Rightarrow x>2$. (It also *makes sense* to say $x>2\Rightarrow x>3$, although that implication statement is false.)

Note: An implication statement $A\Rightarrow B$ means that *if* statement A is true, *then* statement B is true. In particular, if statement A is *false*, then the implication statement $A\Rightarrow B$ is automatically *true*!!! (We sometimes say that $A\Rightarrow B$ is "vacuously true" in this case.)

Converse and Contrapositive. Suppose A and B are statements.

- The *converse* of "A implies B" is "B implies A".
- The *contrapositive* of "A implies B" is "(B is false) implies (A is false)".
  More succinctly, "(not B) implies (not A)".

Thus, the statement "$x>3\Rightarrow x>2$" has **converse**: "$x>2\Rightarrow x>3$," and **contrapositive**: "$x\le 2\Rightarrow x\le 3$."

It is a fact that the contrapositive "(not B) implies (not A)" is *logically equivalent* to "A implies B". (That is, the two implication statements are either both true or both false together.) However, the the truth of *converse* "B implies A" may or may not agree with the truth of "A implies B".

Meanwhile, if we are lucky enough that *both* implications $A\Rightarrow B$ and $B\Rightarrow A$ are true, we write $A\Leftrightarrow B$, and we say "A if and only if B" or "A is equivalent to B."

Sets. A *set* is a collection of objects, which are called *elements* of the set. There is no sense of repetition of elements of a set S: any object x either belongs to S or not, just as a person either belongs to a given club or not. Here are some important sets:

- The set of all *integers* is $\mathbb{Z}=\{\dots,-3,-2,-1,0,1,2,3,\dots\}$.
- The set of all *real numbers* is $\mathbb{R}$.
- The set of all *rational numbers* is $\mathbb{Q}$.
- The set with no elements is $\varnothing$, the *empty set*.

Here is some important notation concerning sets:

- $x\in S$ means x is an element of S. Example: $2\in\mathbb{Z}$.
- $x\notin S$ means x is not an element of S. Example: $\frac{1}{2}\notin\mathbb{Z}$.
- $S\subseteq T$ means that *every* element of S is also an element of T.

  We say that S is a *subset* of T and that T *contains* S. Examples: $\mathbb{Z}\subseteq\mathbb{Q}\subseteq\mathbb{R}$ and $\mathbb{Z}\subseteq\mathbb{Z}$.

- $S \not\subseteq T$ means that S is *not* a subset of T. Equivalently, there is *at least one* element $x \in S$ such that $x \notin T$. Example: $\mathbb{R} \not\subseteq \mathbb{Z}$, because $1/2 \in \mathbb{R}$ but $1/2 \notin \mathbb{Z}$.

- $S = T$ means that $S \subseteq T$ and $T \subseteq S$. Equivalently, S and T consist of exactly the same collection of objects.

- $S \subsetneq T$ means that $S \subseteq T$ and $S \neq T$. We say that S is a *proper subset* of T and that T *properly contains* S. Example: $\mathbb{Z} \subsetneq \mathbb{R}$.

Describing Sets. There are two basic ways to describe a set.

- Listing elements: Some sets can be described by listing all of their elements inside curly brackets $\{$ and $\}$. Example: The set of positive squares is $\{1, 4, 9, 16, \ldots\}$. When listing the elements of a set, order is unimportant, as are repetitions. Thus

  $$\{1,2,3\} = \{3,2,1\} = \{1,1,2,3\} = \{3,2,1,2,3,2,1,2,3\},$$

  since all four sets contain exactly the same elements, namely 1, 2, and 3.

- Set-builder notation: We can sometimes describe a subset by the conditions its elements satisfy. Example: The set of positive real numbers is $\{x \in \mathbb{R} \mid x > 0\}$.

  Important Warning: set-builder notation is only for building *subsets* of sets we already have. So if we already have some set S lying around, we are allowed to define a new set T as

  $$T = \{x \in S \mid \text{ (some condition or other) } \},$$

  but we are *not* allowed to define a set U as

  $$U = \{x \mid \text{ (some condition or other) } \}.$$

Operations on Sets. Here are some other sets that can be created from two sets S and T.

- The *union* $S \cup T$ is the set consisting of all objects x that belong to *either* S *or* T (or both).

  Thus, $x \in S \cup T$ precisely when x lies in *at least one* of the sets S and T. Examples:

  $$\{1,2,3,4\} \cup \{3,4,5,6\} = \{1,2,3,4,5,6\}$$
  $$\{n \in \mathbb{Z} \mid n \geq 0\} \cup \{n \in \mathbb{Z} \mid n < 0\} = \mathbb{Z}.$$

- The *intersection* $S \cap T$ is the set

  $$S \cap T = \{x \in S \mid x \in T\} = \{x \in T \mid x \in S\}.$$

  Thus an element lies in $S \cap T$ precisely when it lies in *both* of the sets. Examples:

  $$\{1,2,3,4\} \cap \{3,4,5,6\} = \{3,4\}$$
  $$\{n \in \mathbb{Z} \mid n \geq 0\} \cap \{n \in \mathbb{Z} \mid n < 0\} = \varnothing.$$

- The *set difference* $S \smallsetminus T$ is the set of elements that are in S but not in T. That is,

  $$S \smallsetminus T = \{x \in S \mid x \notin T\}.$$

  Example: $\{1,2,3,4\} \smallsetminus \{3,4,5,6\} = \{1,2\}$. Please note: as this example shows, T does *not* need to be contained in S in order to talk about $S \smallsetminus T$.

  A common alternative notation for $S \smallsetminus T$ is $S - T$.

Quantifiers. Suppose S is a set, and $P(x)$ is a statement involving the variable x. Assume that the statement $P(x)$ always makes sense for *any* $x \in S$. (Some examples of statements that **don't** make sense are "5 is blue" and "this statement is false." Please note that the problem with "5 is blue" is *not* that it's false, but that we haven't defined what it means for a number to be blue!)

In the situation that $P(x)$ makes sense for any $x \in S$, we can apply the *quantifiers* $\forall$ or $\exists$ to make new statement:

- $\forall x \in S$, $P(x)$, read aloud as, "For all x in S, $P(x)$ [is true]." The symbol $\forall$ means "for all" or "for every" and is called the *universal quantifier*.

  Example: $\forall x \in \mathbb{R}$, $x^2 - 1 = (x+1)(x-1)$.

  (In this example, $P(x)$ is the statement $x^2 - 1 = (x+1)(x-1)$.)

- $\exists x \in S$ s.t. $P(x)$, read aloud as, "There exists some x in S such that $P(x)$ [is true]." The symbol $\exists$ means "there exists" or "there is" and is called the *existential quantifier*.

  Example: $\exists x \in \mathbb{Z}$ s.t. $x > 5$.

Negations of Quantifiers. Please note:

- The statement not$\big(\forall x \in S,\ P(x)\big)$, i.e., the negation of the statement $\forall x \in S$, $P(x)$, is equivalent to $\exists x \in S$ s.t. $\big(\text{not } P(x)\big)$.
- The statement not$\big(\exists x \text{ s.t. } P(x)\big)$, i.e., the negation of the statement $\exists x$ s.t. $P(x)$, is equivalent to $\forall x$, $\big(\text{not } P(x)\big)$.

Make sure you spend enough time thinking about those two equivalences that you can really see why they hold.

General Comments on Writing Proofs. Most of mathematical theorems are of the form

Suppose (some list of hypotheses). Then (some conclusion) must be true.

A proof of that theorem starts from the hypotheses and, via a step-by-step chain of reasoning — with a justification given for each step that isn't essentially obvious — arrives at last at the conclusion. (For Homework 0, by the way, "direct" is one of the secret words.) **BUT**, that logical order is almost never how you **think of** and **design** the proof. Instead, figure out the basic structure of your proof by looking at the **conclusion**; then use the hypotheses to fill in the gaps in the structure. In other words,

Your first thought should **NOT** be, "What do I know from the hypotheses?"

but instead,

Your first thought **SHOULD** be, "Forget the hypotheses; what am I trying to **prove**?"

Examples: (In all the following examples, I'm ignoring any hypotheses.)

- If the desired **conclusion** is $\forall x \in S$, $P(x)$, (i.e., a **for all** proof) then the proof should begin:

  Proof. Given $x \in S$.

  and should end

  so $P(x)$. QED

- If the desired **conclusion** is $S \subseteq T$, which is the same as the statement $\boxed{\forall x \in S, \text{ we have } x \in T}$ then the proof should begin:

  Proof. Given $x \in S$.

  and should end

  so $x \in T$. QED

- If the desired **conclusion** is $S = T$, then remember that that means $S \subseteq T$ **and** $T \subseteq S$. So the proof should come in two chapters. The first chapter should begin

  Proof. ($\subseteq$) Given $x \in S$.

  and should end

  so $x \in T$. QED ($\subseteq$)

  But we're not done yet; immediately on the next line, the second chapter begins

  ($\supseteq$) Given $x \in T$.

  and ends

  so $x \in S$. QED ($\supseteq$). QED Theorem.

- If the desired **conclusion** is $\exists\, x \in S$ s.t. $P(x)$, (i.e., an **existence proof**), then somewhere in the proof — perhaps right at the start, or perhaps somewhere in the middle — you should say,

  Let $x = \boxed{}$,

  where the box is to be filled in with some formula that you'll have to figure out in the side scratchwork that you doodle before you write the formal proof. Then sometime after that — maybe immediately, maybe a little later — show that

  so $x \in S$

  and finally that

  so $P(x)$. QED

  i.e., verify that x actually *does* belong to S, and that the statement $P(x)$ is indeed true.

Once you've realized what the desired structure is, only **then** do you actually look at the hypotheses and see if you can figure out how to fill in all the steps and boxes in the middle.

Proof by Contradiction. Sometimes there may not be a direct proof (i.e., following structures like those described above). Another option is a proof by contradiction. This style of proof begins,

Proof. Suppose not. Then…

That is, suppose that the desired conclusion were not actually true. The proof would ultimately end by arriving at some obviously contradictory statement (like $x = 0$ and also $x \neq 0$) and then declaring:

Contradiction! QED

(That is, our supposition led to something impossible. So our supposition must be false; that is, the desired statement is actually true.)

Warning. Don't overuse proofs by contradiction; all too often they are clunky and unnecessarily complicated. You should try direct proofs before you resort to trying a proof by contradiction.

Proof by Induction. If the statement you have is of the form, "For all integers $n \geq 1$, $P(n)$," (where $P(n)$ is some property that may be true or false for any such integer), it may be best to use mathematical induction. See Section 0 of Saracino's book, and the review video on induction, for more information.