Left and Right Cosets

In this handout, we'll prove the following result:

Theorem. Let G be a group, and let $H \subseteq G$ be a subgroup. Define

$$R = \{ \text{ right cosets of } H \text{ in } G \} \quad \text{and} \quad L = \{ \text{ left cosets of } H \text{ in } G \}$$

Then $|R| = |L|$.

That is, a subgroup always has the same number of right cosets as left cosets.

Note: If G is a *finite* group, then we can prove the above theorem using the ideas from Lagrange's Theorem. Specifically, if G is finite we can write the distinct right cosets of H as

$$Ha_1,\ Ha_2,\ \cdots,\ Ha_k,$$

for some $a_1, \ldots, a_k \in G$, so that $|R| = k$. But we can also write the left cosets of H as

$$b_1H,\ b_2H,\ \cdots,\ b_\ell H,$$

for some $b_1, \ldots, b_\ell \in G$, so that $|L| = \ell$.
Writing $|G| = n$ and $|H| = m$, we have $|Ha_i| = |H| = m$ and $|b_jH| = |H| = m$ for every i, j, so we have

$$km = |Ha_1| + \cdots + |Ha_k| = |G| = |b_1H| + \cdots + |Hb_\ell| = \ell m.$$

Therefore, since $m < \infty$, we may divide both sides by m to obtain $k = \ell$, i.e., $|R| = |L|$.

However, the above proof only works if G is finite, and the theorem we wish to prove needs to apply to *all* groups G. So the rest of this handout is devoted to the fully general proof, which is actually a little simpler than the above proof!

Proof of Theorem. Define

$$\boxed{f : R \to L \qquad \text{by} \qquad Ha \mapsto a^{-1}H}$$

It suffices to show that f is a bijective function. [Recall that this is what it means to say two sets have the same cardinality: that there is a bijective function from one to the other.]

Function/Well-Defined: It's not even immediately obvious that f is a function, so we start with that.
More precisely, any element of R is indeed a right coset and hence of the form Ha for some $a \in G$, and therefore $f(Ha) = a^{-1}H$ is indeed a left coset. That is to say, f is *defined*, but we also need to show that f is *well-defined*, since there is usually more than one way to write any given coset.

That is, given a right coset $Ha = Hb \in R$ written two ways — i.e., Ha and Hb are the same right coset even though a and b might be different — we need to prove that $f(Ha) = f(Hb)$. That is, we need to prove that $a^{-1}H = b^{-1}H$.

Well, since $Ha = Hb$, we have $ab^{-1} \in H$ by the right coset relation. Rewriting $a = (a^{-1})^{-1}$, then,we have $(a^{-1})^{-1}b^{-1} \in H$. But this is the *left* coset relation for a^{-1} and b^{-1}, meaning that $a^{-1}H = b^{-1}H$. That is, $(Ha) = f(Hb)$, as desired.

One-to-one: Given $a, b \in G$ with $f(Ha) = f(Hb)$, then $a^{-1}H = b^{-1}H$, by definition of f. Therefore, $(a^{-1})^{-1}b^{-1} \in H$, by the left coset relation. That is, $ab^{-1} \in H$, so $Ha = Hb$ by the right coset relation, as desired.

Onto: Given a left coset $bH \in L$, then $b \in G$, and hence $Hb^{-1} \in R$ is a right coset.
We have $f(Hb^{-1}) = (b^{-1})^{-1}H = bH$, as desired. QED

Note 1. In the "One-to-One" portion of the proof, please notice that we were given two elements of R (namely Ha and Hb) for which $f(Ha) = f(Hb)$, and our job was to prove that $Ha = Hb$, i.e., that the original two elements of R were already equal **as elements of** R.

In particular, our job was *not* to prove the (probably false) statement that $a = b$; we just needed to prove that the right coset Ha was equal to the right coset Hb, whether or not a and b themselves were actually equal.

Note 2. Did it strike you as funny that the formula for f was $f(Ha) = a^{-1}H$? Why not use a different, simpler formula, like $F(Ha) = aH$?

To answer that question, it's worth going back through the proof to see what would go wrong if you tried to use $F : R \to L$ by $F(Ha) = aH$. And the answer is: **that's not even a function**, because it isn't even well-defined.

That is, it's sometimes possible that $Ha = Hb$ but $aH \neq bH$, and hence that we would have $F(Ha) \neq F(Hb)$. That would be bad; a function can't spit out different actual outputs from the same input just because you wrote the input in a slightly different way.

Example. Let $G = S_3$ and let $H = \langle(1, 2)\rangle = \{e, (1, 2)\}$.
Let $a = (2, 3)$ and $b = (1, 2, 3)$. Then $Ha = \{(2, 3),\ (1, 2, 3)\} = Hb$
[which you can also see by the right coset relation, since $ab^{-1} = (2, 3)(1, 3, 2) = (1, 2) \in H$].
BUT $aH = \{(2, 3),\ (1, 3, 2)\} \neq \{(1, 2, 3), (1, 3)\} = bH$
[which you can also see by the left coset relation, since $a^{-1}b = (2, 3)(1, 2, 3) = (1, 3) \notin H$].

The above example shows that the formula $F(Ha) = aH$ is **not even a function** from R to L, because it is not well-defined.