Math 350, Spring 2026 Professor Rob Benedetto

Optional Handout: Group Actions

§1. Definition and Examples

Definition. Let G be a group, and let X be a set. An ***action*** of G on X is a function

$$G \times X \longrightarrow X$$

where $(g, x) \in G \times X$ maps to an element of X that we will denote $g \cdot x \in X$, in such a way that the following properties hold:

1. $e \cdot x = x$ for all $x \in X$.
2. $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$ and $x \in X$.

Idea: To put it another way, when a group G acts on a set X, what is happening is that every element $g \in G$ becomes a function $g : X \to X$; the formula for g as a function is $g(x) = g \cdot x$. Note: condition (1) of the definition says that $e \in G$ is the identity function $\mathrm{id}_X : X \to X$. And then condition (2) says that multiplication of elements of G becomes composition of functions; after all, for any $x \in X$, we have

$$g \circ h(x) = g(h(x)) = g(h \cdot x) = g \cdot (h \cdot x) = (gh) \cdot x = (gh)(x),$$

where the second-to-last equality is precisely condition (2).
Combining those two facts, then, for each $g \in G$, the function $g : X \to X$ **must be one-to-one and onto**. After all, if $g^{-1} \in G$ is the inverse of g as an element of G, then

$$g \circ g^{-1} = (gg^{-1}) = e = \mathrm{id}_X \quad \text{and} \quad g^{-1} \circ g = (g^{-1}g) = e = \mathrm{id}_X,$$

where in each case, the first equality is by condition (2), and the last is by condition (1). The punchline here is that the *group* inverse $g^{-1} \in G$, when considered as a function from X to X, is the inverse of g *as a function*.

Example. $G = GL(n, \mathbb{R})$ acts on $X = \mathbb{R}^n$ by matrix multiplication: $A \in GL(n, \mathbb{R})$ and $v \in \mathbb{R}^n$ gives $A \cdot v = Av$. This is a group action, since

- $I_n \cdot v = I_n v = v$ for every $v \in \mathbb{R}^n$ since I_n is the identity matrix.
- $A \cdot (B \cdot v) = A(Bv) = (AB)v = (AB) \cdot v$ since matrix multiplication is associative.

Put another way, each matrix $A \in GL(n, \mathbb{R})$ is effectively becoming a function here: A becomes the function $A : \mathbb{R}^n \to \mathbb{R}^n$ by $v \mapsto Av$. Moreover, as you learned in linear algebra, the group operation of matrix multiplication AB becomes composition of functions, precisely because matrix multiplication is associative: $(AB)v = A(Bv)$.

Example. The symmetric group $G = S_n$ acts on $X = \{1, \dots, n\}$ by $\sigma \cdot i = \sigma(i)$. This is a case in which the group action comes directly from the definition of the group; that is, S_n is *defined* to be the set of one-to-one and onto functions from X to itself, and the group operation is *defined* to be composition.

Please note that any *subgroup* of S_n, like A_n or D_n, also acts on $\{1, \ldots, n\}$. Similarly, any subgroup of $GL(n, \mathbb{R})$ acts on $\mathbb{R}^n$. And in general, if G acts on X and H is a subgroup of G, then H acts on X.

Example. Let H be a subgroup of a group G. Then H (viewed as a group) acts on G (viewed as a set) by

$$h \cdot g = hg \quad \text{for} \quad h \in H \text{ and } g \in G.$$

You should check that this function $H \times G \to G$ is indeed a group action (of the group H acting on the set G).

Example. Let H be a subgroup of a group G, and let $L = \{gH : g \in G\}$ be the set of *left* cosets of H in G. Then G (viewed as a group) acts on L (viewed as a set), by

$$g \cdot (aH) = (ga)H \quad \text{for } g \in G \text{ and } aH \in L.$$

You should check that this function $G \times L \to L$ is well-defined and is a group action (of the group G acting on the set L).

Example. Let G be a group. Then G acts on itself by conjugacy, i.e., by

$$g \cdot h = ghg^{-1} \quad \text{for} \quad g, h \in G.$$

You should check that this function $G \times G \to G$ is a group action (of the group G acting on the set G).

Example. Let $C \subseteq \mathbb{R}^3$ be the cube of side-length 2 centered at the origin and with faces parallel to the coordinate planes. That is:

$$C = [-1, 1] \times [-1, 1] \times [-1, 1] \subseteq \mathbb{R}^3.$$

Let G be the set of all rotations about the origin that take C to itself. One can check that G is a group under composition. We call G the ***rotation group*** of the cube.
I sketched a proof in class that this group G is isomorphic to S_4, by showing that G acts on the four-element set of diagonals of C.

§2. Orbits

Definition. Let a group G act on a set X. Then the ***orbit*** of $x \in X$ is the set

$$G \cdot x = \{g \cdot x \mid g \in G\}.$$

Warning: An orbit is a sub**set** of the **set** X. In particular, it is not a group, subgroup, coset, or anything like that.

Example. Consider the group of all rotations of the plane about the origin. One can prove that this is the group of rotation matrices

$$SO(2, \mathbb{R}) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R} \right\}.$$

The rotation group $SO(2, \mathbb{R})$ acts on $\mathbb{R}^2$ by matrix multiplication. Given a nonzero vector $v \in \mathbb{R}^2$ of length r, you should try to prove that the orbit of v is

$$SO(2, \mathbb{R}) \cdot v = \text{the circle of radius } r \text{ centered at the origin.}$$

If we think the origin as the sun and v as a planet, then this circle is a simple model of the astronomical orbit of the planet; this is why the term "orbit" is used for the mathematical concept here. (On the other hand, what is the orbit of the origin $(0,0)$?)

But don't take the planetary orbit idea too far; I was just explaining where the word comes from. More generally, the orbit of $x \in X$ is just the set of everything x gets mapped to by all the functions $g : X \to X$ in the group G.

Example. When a subgroup H of a group G acts on G via $h \cdot g = hg$, the orbit of $g \in G$ is $H \cdot g = Hg$, which is the right coset of g.

Example. Consider the rotation group G of the cube C. Here are some interesting orbits:

- Any rotation in G takes a vertex to a vertex. If v is one vertex, then

$$G \cdot v = \{\text{the 8 vertices of } C\}.$$

- Any rotation in G takes an edge midpoint to an edge midpoint. If m is one edge midpoint, then

$$G \cdot m = \{\text{the 12 edge midpoints of } C\}.$$

- Any rotation in G takes a face center to a face center. If c is one face center, then

$$G \cdot c = \{\text{the 6 face centers of } C\}.$$

Example. When S_n acts on $\{1, \ldots, n\}$, there is only one orbit. In other words, for any $i \in \{1, \ldots, n\}$,

$$S_n \cdot i = \{1, \ldots, n\}.$$

This follows because for any $j \in \{1, \ldots, n\}$, there is $\sigma \in S_n$ with $\sigma(i) = j$. Do you see why?

In general, the orbits of G acting on X are called ***G-orbits***. Here is an important result.

Theorem. If G acts on X, then any two G-orbits are either equal or disjoint. In particular, the set of G-orbits forms a partition of X.

Proof. (Sketch.) Define a relation $\sim$ on X by the following rule: given $x, y \in X$, we define $x \sim y$ to mean that there exists $g \in G$ such that $x = g \cdot y$. I will leave it to you to prove that this relation $\sim$ is in fact an equivalence relation on X. (Details LTR.)

I will also leave it to you to prove, for any $x \in X$, that the equivalence class $[x]_\sim$ of x is precisely the orbit $G \cdot x$. (That is, show that the set of elements equivalent to x under $\sim$ is equal to the set of elements of the form $g \cdot x$. Again, details LTR.)

Then we are done by the properties of equivalence classes. QED

Example. In our earlier example of the rotation group $SO(2, \mathbb{R})$ acting on the plane $\mathbb{R}^2$, we saw that the orbits are the circles C_r of radius $r > 0$ centered at the origin together with the origin itself. Thus the partition induced by $SO(2, \mathbb{R})$-orbits is

$$\mathbb{R}^2 = \{(0,0)\} \cup \bigcup_{r>0} C_r.$$

Example. When a group G acts on itself by conjugacy (recall: this action is given by the formula $g \cdot h = ghg^{-1}$), the G-orbit $G \cdot h$ is called the ***conjugacy class*** of h, which I've been denoting by $[h]_{\text{conj}}$, but there are a lot of different notations for it out there. Thus,

$$[h]_{\text{conj}} = \{ghg^{-1} \mid g \in G\}.$$

You showed in Exercise 9.12 that conjugacy is an equivalence relation on G, and thus that conjugacy classes form a partition of G. That is, you already verified the proof of the Theorem above in this case.

Example. Fix a permutation $\sigma \in S_n$, and instead of considering the action of the full group S_n on $X = \{1, \ldots, n\}$, consider only the action of the cyclic subgroup $\langle \sigma \rangle \subseteq S_n$ on $X = \{1, \ldots, n\}$ — that is, the action is given by $\sigma^k \cdot i = \sigma^k(i)$ for all $k \in \mathbb{Z}$ and $i \in X$.
Now write

$$\sigma = (a_{11} \ \cdots \ a_{1k_1})(a_{21} \ \cdots \ a_{2k_2}) \ \cdots (a_{m1} \ \cdots \ a_{mk_m})$$

as a product of disjoint cycles. Then one can show that

$$\langle \sigma \rangle \cdot a_{11} = \{a_{11}, \ldots, a_{1k_1}\}, \ \langle \sigma \rangle \cdot a_{21} = \{a_{21}, \ldots, a_{2k_2}\}, \text{ etc.}$$

(See if you can prove this!) Thus, the disjoint cycles making up σ determine the orbits of $\langle \sigma \rangle$ acting on $\{1, \ldots, n\}$.

§3. Stabilizers

When a group G acts on a set X, an element $x \in X$ gives two interesting sets. The first is the orbit $G \cdot x = \{g \cdot x \mid g \in G\}$ studied in the previous section. The second interesting set is the following subgroup of G.

Definition. Given an action of G on X and an element $x \in X$, the ***stabilizer*** of x, or ***isotropy subgroup*** of x, is

$$G_x = \{g \in G \mid g \cdot x = x\} \subseteq G.$$

Proposition. G_x is a subgroup of G.

Proof. LTR. If you attempt none of the "LTR" proofs on this handout, *try this one.* It is essentially the same proof as you would have needed to do for Exercise 8.16. QED

Example. When G acts on itself by conjugation, the stabilizer of $g \in G$ is the subgroup

$$\{h \in G \mid g \cdot h = h\} = \{h \in G \mid ghg^{-1} = h\} = \{h \in G \mid gh = hg\}.$$

This is the ***centralizer*** $Z(g)$ of g. In Exercise 5.23, you proved that $Z(g)$ was a subgroup of G. We get a second proof of this fact since centralizers are stabilizers for the conjugacy action.

Example. For the rotation group G of the cube C, here are some interesting stabilizers:

- If v is a vertex, then consider the diagonal d through the center of the cube that connects v to its opposite vertex $-v$. Then any rotation about d fixes v, and

  $$G_v = \{e, \text{rotate } 120^\circ \text{ about } d, \text{rotate } 240^\circ \text{ about } d\}.$$

  So G_v has three elements. Recall from §2 that $|G \cdot v| = 8$.

- If m is the edge midpoint of an edge E, then m is fixed by the 180° rotation of the cube that takes E to itself. Hence

$$G_m = \{e, \text{the } 180^\circ \text{ rotation that takes } E \text{ to itself}\}.$$

So G_m has two elements. Recall from §2 that $|G \cdot m| = 12$.

- If c is the center of a face F, then c is fixed when we rotate about the line perpendicular to F through c. Thus

$$G_c = \{e, \text{rotate } 90^\circ \text{ about } F, \text{rotate } 180^\circ \text{ about } F, \text{rotate } 270^\circ \text{ about } F\}.$$

So G_c has four elements. Recall from §2 that $|G \cdot c| = 6$.

I encourage you to play with a cube so you can see these stabilizers. Also note that

$$\begin{aligned} |G_v| \cdot |G \cdot v| &= 3 \cdot 8 = 24 \\ |G_m| \cdot |G \cdot m| &= 2 \cdot 12 = 24 \\ |G_c| \cdot |G \cdot c| &= 4 \cdot 6 = 24. \end{aligned} \tag{1}$$

We will see in the next section that this is no accident.

Example. Consider the action of S_n on the set $X = \{1, \ldots, n\}$. The isotropy subgroup of $n \in X$ is

$$(S_n)_n = \{\sigma \in S_n \mid \sigma(n) = n\}.$$

Each such σ fixes n and permutes $1, \ldots, n-1$, so that the stabilizer is a copy of S_{n-1}. (More precisely, $(S_n)_n$ is isomorphic to S_{n-1}.) In §3, we saw that the orbit of n is $S_n \cdot n = \{1, \ldots, n\}$. It follows that

$$|(S_n)_n| \cdot |S_n \cdot n| = |S_{n-1}| \cdot |\{1, \ldots, n\}| = (n-1)! \cdot n = n! = |S_n|.$$

This is, again, a special case of the main result of the next section.

§4. The Orbit-Stabilizer Theorem

Orbit-Stabilizer Theorem. Let G be a group acting on a set X, and let $x \in X$. Recall that the set $G \cdot x \subseteq X$ is the orbit of x, and the subgroup $G_x \subseteq G$ is the stabilizer of G. Then

$$[G : G_x] = |G \cdot x|.$$

That is, the number of left cosets of G_x in G exactly equals the size of the orbit of x.

Proof. Let $L = \{gG_x : g \in G\}$ denote the set of left cosets of G_x in G. Define a function $f : L \to G \cdot x$ by the formula

$$f(gG_x) = g \cdot x.$$

It suffices to show that f is well-defined, one-to-one, and onto.

Well-defined: Given $g, h \in G$ such that $gG_x = hG_x$, then $h^{-1}g \in G_x$.
Since $g = (hh^{-1})g = h(h^{-1}g)$, we obtain

$$f(gG_x) = g \cdot x = \big(h(h^{-1}g)\big) \cdot x = h \cdot \big((h^{-1}g) \cdot x\big) = h \cdot x = f(hG_x),$$

as desired, where second equality uses the definition of group action and the third equality uses the fact that $h^{-1}g \in G_x$.

One-to-one. Given $g, h \in G$ such that $f(gG_x) = f(hG_x)$, then $g \cdot x = h \cdot x$ by the definition of f. Hence

$$(h^{-1}g) \cdot x = h^{-1} \cdot (g \cdot x) = h^{-1} \cdot (h \cdot x) = (h^{-1}h) \cdot x = e \cdot x = x.$$

(Note: the first, third and fifth equalities use the definition of group action.) Since $(h^{-1}g) \cdot x = x$, we have $h^{-1}g \in G_x$, by definition of stabilizer. Thus, $gG_x = hG_x$, as desired.

Onto. Given $y \in G \cdot x$, then by the definition of orbit, there is some $g \in G$ such that $y = g \cdot x$. Then $f(gG_x) = g \cdot x = y$, as desired. QED

Big Corollary. (Often *also* called the Orbit-Stabilizer Theorem.)
Let G be a *finite* group acting on a set X, and let $x \in X$. Then

$$|G| = |G_x| \cdot |G \cdot x|.$$

Proof. Since G is finite, we have

$$|G| = |G_x| \cdot [G : G_x] = |G_x| \cdot |G \cdot x|,$$

where the first equality is by Lagrange's Theorem, and the second is by the Orbit-Stabilizer Theorem. QED

Example. Let G be the rotation group of the cube. Then the Corollary and the computations (1) on page 5 of this handout confirm our in-class theorem that G is isomorphic to S_4, since $3 \cdot 8 = 2 \cdot 12 = 4 \cdot 6 = 24 = |S_4|$.

Example Let $G = D_4$ be the dihedral group of order 8 (i.e., the set of rotations and flips of a square), and let X be the set of all ways to color each of the four sides of a square with the colors red, yellow, and blue. (So $|X| = 3^4 = 81$, because there are three choice of color for each of the four sides.) Note that D_4 acts on X by rotating or flipping the square, and the colored sides with it.
Consider the element $x \in X$ corresponding to coloring the top and bottom side red, and the other two sides blue. No matter how you rotate or flip this square, it will always result in one of two arrangements: either top and bottom red with vertical sides blue, or top and bottom blue with vertical sides red. That is, $|D_4 \cdot x| = 2$; there are two elements of the orbit. Since $|D_4| = 8$, it follows by the Orbit-Stabilizer Theorem that there are exactly four rotations or flips of x that leave the coloring the same. (It's not hard to check that they are e, f^2, fg, and f^3g, in the book's notation on pages 74–75. Incidentally, this four-element stabilizer subgroup is isomorphic to V_4.)
On the other hand, consider the element $y \in X$ corresponding to coloring the top and bottom side red, the left side blue, and the right side yellow. Now there are four possible arrangements, because there are four choices for where the unique blue side goes. (Once you know where the blue side goes, the opposite side *must* be yellow, and the two adjacent sides *must* be red.) Thus, $|D_4 \cdot y| = 4$, and so by the Orbit-Stabilizer Theorem, there are exactly two rotations or flips of y that leave the coloring the same. (It's easy to see that they are e and f^3g; this two-element stabilizer subgroup is isomorphic to C_2.)