Functions: Fundamental Definitions and Facts

This handout should be mostly review, but some of it may be new to you. It covers much of the material in Section 7 of Saracino.

Definition. Let S, T be sets. A **function** $f : S \to T$ assigns, to each $s \in S$, a unique $t \in T$. In that case, we write $f(s) = t$, or $f : s \mapsto t$. That is, $\boxed{\forall\, s \in S,\ \exists!\, t \in T \text{ s.t. } f(s) = t}$

The set S is called the **domain** of f, and the set T is called the **codomain** or **target set** of f.

(**Warning**: do *not* call T the "range" of T, as that means something different; see below.)

Actually, the above definition is a little informal. See Video 17 for a fully rigorous definition; but in practice, we will use the above informal definition, which is more intuitive.

For both definitions, the $\exists!$ statement — i.e., "there exists a unique" — in the box above is what really matters. To clarify things in practice, if we have a rule $f : S \to T$ which were are not yet certain is truly a function, we sometimes use the following terminology:

- To say $f : S \to T$ is **defined** means $\boxed{\forall\, s \in S,\ \exists t \in T \text{ s.t. } f(s) = t}$
- To say $f : S \to T$ is **well-defined** means $\boxed{\forall\, s_1, s_2 \in S,\ \text{ if } s_1 = s_2,\ \text{ then } f(s_1) = f(s_2)}$

So a function, then, is a rule/recipe $f : S \to T$ that is **both** defined and well-defined.

Example: $f : \mathbb{Z} \to \mathbb{Q}$ by $f(x) = \dfrac{1}{x}$ is **not defined**
because $0 \in \mathbb{Z}$ is in the (supposed) domain, but $f(0) = 1/0$ doesn't make sense.

Example: $f : \mathbb{Q} \to \mathbb{Z}$ by $f\left(\dfrac{m}{n}\right) = m$ is **not well-defined**
because $\dfrac{3}{5} = \dfrac{6}{10}$ but $f\left(\dfrac{3}{5}\right) = 3 \neq 6 = f\left(\dfrac{6}{10}\right)$.

Definition. Let S, T be sets, and let $f : S \to T$ be a function.

$\boxed{1}$ The set $\{f(s) \mid s \in S\} = \{t \in T \mid \exists\, s \in S \text{ s.t. } f(s) = t\}$, which is a subset of T, is called the **image** of f (or the **range** of f) and is denoted $f(S)$ or $\operatorname{im}(f)$.

$\boxed{2}$ We say f is **onto**, or **surjective**, if $f(S) = T$, i.e., if $\boxed{\forall t \in T, \exists s \in S \text{ s.t. } f(s) = t}$

$\boxed{3}$ We say f is **one-to-one**, or **injective**, if $\boxed{\forall s_1, s_2 \in S \text{ s.t. } f(s_1) = f(s_2), \text{ we have } s_1 = s_2}$

$\boxed{4}$ We say f is **bijective** if it is both injective and surjective.

$\boxed{5}$ The **identity function** on S is the function $\mathrm{id}_S : S \to S$ given by $\mathrm{id}_S(x) = x$ for all $x \in S$.

Note that the identity function $\mathrm{id}_S : S \to S$, sometimes denoted i_S or 1_S, *is* indeed a function, and it is bijective.

Note also both the similarities **and** the differences between:

- the definitions of "f is defined" and "f is onto"
- the definitions of "f is well-defined" and "f is one-to-one."

In particular, note that for well-defined, the implication goes one direction, but for one-to-one, it goes the other direction.

What does it mean to say $f_1 = f_2$? It's always important in mathematics to know *precisely* what we mean when we say two objects are equal.
For functions: if f_1 and f_2 are functions, we say/write $f_1 = f_2$ if:

- the domain S of f_1 is the same as the domain of f_2 (in the sense of equality of sets), and
- for all $s \in S$ [i.e., for all s in this common domain], we have $f_1(s) = f_2(s)$.

Definition. Let S, T, U be sets, and let $f : S \to T$ and $g : T \to U$ be functions.
In this situation — where the domain of g is the same as the target set of f — we define the **composition** of g and f, denoted $g \circ f$, to be the function

$$\boxed{g \circ f : S \to U \quad \text{given by} \quad g \circ f(s) = g(f(s))}$$

We may read $g \circ f$ aloud as "g composed with f" or "g of f" or "g after f."

Definition. Let S, T be sets, and let $f : S \to T$ be a function. We say f is **invertible** if there exists a function $g : T \to S$ such that

$$\boxed{f \circ g = \mathrm{id}_T \text{ and } g \circ f = \mathrm{id}_S}$$

that is, if both:

- for all $s \in S$, we have $g(f(s)) = s$, and
- for all $t \in T$, we have $f(g(t)) = t$.

In that case, g is called the **inverse function** of f, or simply the **inverse** of f, and we write $f^{-1} = g$.

The following result is standard, but I recommend you try to prove it yourself; it is good practice both with writing if-and-only-if proofs, and with working with the various definitions in this handout. (The informal discussion on page 62 of Saracino may also be helpful.)

Proposition. Let S, T be sets, and let $f : S \to T$ be a function. Then

f is invertible **if and only if** f is bijective.

Moreover, in that case, the inverse function $f^{-1} : T \to S$ is given by:

$$f^{-1}(t) = \text{the unique } s \in S \text{ such that } f(s) = t$$

for all $t \in T$.

Here's another standard result about inverses. Again, I encourage you to try to prove it yourself. (It may remind you of Theorems 3.2, 3.3, and 3.4 about groups!)

Proposition. Let S, T, U be sets, and let $f : S \to T$ and $g : T \to U$ be **invertible** functions.

1 The inverse of f is unique. That is, if $F : T \to S$ is a function such that $F \circ f = \mathrm{id}_S$ and $f \circ F = \mathrm{id}_T$, then we have $F = f^{-1}$.

2 The inverse function f^{-1} is also invertible, and its inverse is $(f^{-1})^{-1} = f$.

3 The composition $g \circ f$ is also invertible, and its inverse is $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Finally, composition of functions is associative; again, try to prove the following result yourself:

Proposition. Let S, T, U, V be sets, and let $f : S \to T$, $g : T \to U$, and $h : U \to V$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

[Note: $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are functions from S to V. Proving they are equal means proving that for all $s \in S$, their values at s agree.]