Solutions to Homework #9

1. Saracino, Section 8, Problem 8.2: Write each permutation as a product of disjoint cycles, and then as a product of transpositions. Determine whether each permutation is even or odd.

(a): $\begin{pmatrix}1&2&3&4&5&6\\3&6&1&4&2&5\end{pmatrix}$ (b): $\begin{pmatrix}1&2&3&4&5&6\\2&4&6&1&3&5\end{pmatrix}$

(c): $\begin{pmatrix}1&2&3&4&5&6\\5&6&3&4&1&2\end{pmatrix}$ (d): $\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}$

Solutions.

(a): Direct computation gives: $\boxed{(1,3)(2,6,5)}$ As a product of transpositions this is $\boxed{(1,3)(2,6)(5,6)}$ and so is $\boxed{\text{odd}}$ because there are three transpositions.

(b): Direct computation gives: $\boxed{(1,2,4)(3,6,5)}$ As a product of transpositions this is $\boxed{(1,2)(2,4)(3,6)(5,6)}$ and so is $\boxed{\text{even}}$ because there are four transpositions.

(c): Direct computation gives: $\boxed{(1,5)(2,6)}$ As a product of transpositions this is (already) $\boxed{(1,5)(2,6)}$ and so is $\boxed{\text{even}}$ because there are two transpositions.

(d): Direct computation gives: $\boxed{(1,6,3,4)(2,5)}$ As a product of transpositions this is $\boxed{(1,6)(3,6)(3,4)(2,5)}$ and so is $\boxed{\text{even}}$ because there are four transpositions.

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2. Saracino, Section 8, Problem 8.5: Write down all the elements of S_4 [in disjoint cycle notation], and indicate which ones are in A_4. Check your results against Theorem 8.5.

Solution. First, there's the identity element $\boxed{e, \text{ which IS in } A_4}$

Second, there are the 2-cycles, which are odd and hence not in A_4. There are $\binom{4}{2} = 6$ of these (since we must pick two of the four symbols to be switched), namely

$\boxed{(1,2),\ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,4), \text{ which are NOT in } A_4}$

Third, there are the 3-cycles, which are even and hence in A_4. There are $\binom{4}{3} = 4$ ways to choose which 3 symbols a, b, d will be rotated, and two ways to rotate them, i.e., either (a, b, c) or (a, c, b), for a total of $4 \cdot 2 = 8$. They are:

$\boxed{(1,2,3),\ (1,3,2),\ (1,2,4),\ (1,4,2),\ (1,3,4),\ (1,4,3),\ (2,3,4),\ (2,4,3), \text{ all eight of which ARE in } A_4}$

Fourth, there are the 4-cycles, which are odd and hence not in A_4. Each involves all four symbols $1, 2, 3, 4$, so we can list 1 in the cycle; then we have 3 choices for the next symbol, two for the next after that, and one for the last, for a total of $3! = 6$ such cycles. They are:

$\boxed{(1,2,3,4),\ (1,2,4,3),\ (1,3,2,4),\ (1,3,4,2),\ (1,4,2,3),\ (1,4,3,2), \text{ which are NOT in } A_4}$

Fifth, there are permutations that are two disjoint 2-cycles, of which there are 3, since each such permutation σ is determined by whether $\sigma(1)$ is 2, 3, or 4. So they are:

$\boxed{(1,2)(3,4),\ (1,3)(2,4),\ (1,4)(2,3) \text{ which ARE in } A_4}$

Together, that's $1 + 6 + 8 + 6 + 3 = 24$ elements of S_4, and $1 + 8 + 3 = 12$ elements of A_4. Sure enough, $4! = 24$ and $4!/2 = 12$, agreeing with Theorem 8.5.

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3. Saracino, Section 8, Problems 8.10(b) and 8.3(c):

Let $\sigma = \begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12\\6&7&5&9&8&4&11&3&1&12&2&10\end{pmatrix} \in S_{12}$.

(a): Find the order of σ. [This is problem 8.10(b).]

(b): Decide whether σ is even or odd. [This is problem 8.3(c).]

Solution. In cycle notation, we compute $\sigma = (1,6,4,9)(2,7,11)(3,5,8)(10,12)$.

(a): By the previous problem, $o(\sigma) = \text{lcm}(4,3,3,2) = \boxed{12}$

(b): The 4-cycle is odd, the two 3-cycles are each even, and the 2-cycle is odd. Thus, σ itself is odd + even + even + odd = $\boxed{\text{even}}$

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4. Saracino, Section 8, Problem 8.11(a):
Give an example of two elements $f, g \in S_9$ such that $o(f) = o(g) = 5$ and $o(fg) = 9$.

Solution. [Note: there are LOTS of ways to do this; here is one.]
Let $\boxed{f = (1,2,3,4,5) \text{ and } g = (5,6,7,8,9)}$ both of which have order 5, since they are 5-cycles. Computation gives $fg = (1,2,3,4,5,6,7,8,9)$, which has order 9, since it is a 9-cycle.

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5. Saracino, Section 8, Problem 8.12: Does A_6 have an element of order 6? Does A_7? [As always, prove your answers.]

Solution. For A_6: $\boxed{\text{NO, there is no element of order 6 in } A_6}$
Given $\sigma \in A_6$, write σ in disjoint cycle notation.
If σ is a single cycle, then it has odd length, since it is an even permutation. But the order of a cycle is its length, so $o(\sigma) \neq 6$.
If σ consists of two disjoint cycles, then either both are of odd length or both are of even length, for σ to be even. Thus, the two cycle-lengths are either 2 and 2, 2 and 4, or 3 and 3, which means $o(\sigma)$ is 2, 4, or 3, respectively. Again, $o(\sigma) \neq 6$.
If σ consists of three or more disjoint cycles, then since each must have length at least 2, the only possibility is 3 disjoint 2-cycles. But then σ would be odd, a contradiction; so this doesn't happen.
Thus, $o(\sigma) \neq 6$. QED

For A_7: $\boxed{\text{YES, } A_7 \text{ has elements of order 6}}$
For example, let $\sigma = (1,2)(3,4)(5,6,7)$. Then σ is even (since the two transpositions are each odd, and the 3-cycle is even), so $\sigma \in A_7$. Moreover, $o(\sigma) = \text{lcm}(2,2,3) = 6$. QED

Note: For part (a), I don't expect you to have given a fully detailed proof that no element of A_6 has order 6, but I *do* expect you to show that you really thought through all the possibilities.

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6. Saracino, Section 8, Problem 8.16:
Let X be a set and let $Y \subseteq X$. Prove that $\{f \in S_X \mid \forall y \in Y, \text{ we have } f(y) = y\}$ is a subgroup of S_X.

Proof. Let $H = \{f \in S_X \mid \forall y \in Y, \text{ we have } f(y) = y\}$.
Nonempty: Let $e = \text{id}_X$, the identity function on X. Then $e \in S_X$, and for any $y \in Y$, we have $y \in X$, so $e(y) = y$. So $e \in H$.

Closure: Given $f, g \in H$, then $f \circ g \in S_X$, and for any $y \in Y$, we have

$$f \circ g(y) = f(g(y)) = f(y) = y,$$

so $f \circ g \in H$.

Inverses: Given $f \in H$, then $f^{-1} \in S_X$, and for any $y \in Y$, we have

$$f^{-1}(y) = y \quad \text{because} \quad f(y) = y$$

so $f^{-1} \in H$. QED