Solutions to Homework #8

1. Saracino, Section 7, Problem 7.1(e,g,i): In each example below f is given either as a rule or as a set of ordered pairs. In each case, determine whether f is a function from S to T. When it is, determine whether it is one-to-one, and whether it is onto.

e. $S = T = \mathbb{R}$, $f(x) = x^3$

g. $S = T = \mathbb{R}$, $f(x) = 1/x$

i. $S = T = \mathbb{Z}_{\geq 1}$, $f(x) = \begin{cases} 1 & \text{if } x = 1, \\ x - 1 & \text{if } x > 1 \end{cases}$

(e): YES, FUNCTION because for each $x \in \mathbb{R}$, $f(x) = x^3$ is defined and lies in $\mathbb{R}$.

YES, ONE-TO-ONE because: given $x_1, x_2 \in \mathbb{R}$ such that $x_1^3 = x_2^3$, then we may take cube roots of both sides [which makes sense for any real number], to get $x_1 = x_2$. [Alternatively, use the fact from calculus that $f(x) = x^3$ is strictly increasing, so that if $x_2 > x_1$, then $x_2^3 > x_1^3$, a contradiction.]

YES, ONTO because given $y \in \mathbb{R}$, then y has a (real) cube root $x = \sqrt[3]{y} \in \mathbb{R}$ [even if y is negative], so that $f(x) = x^3 = y$.

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(g): NO, NOT FUNCTION because $f(x) = 1/x$ has $f(0)$ undefined.

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(i): YES, FUNCTION because for each $n \in S$, we either have $n = 1$ (so $f(n) = 1 \in T$), or we have $n \geq 2$ (so $f(n) = n - 1 \in T$).

NO, NOT ONE-TO-ONE because $f(1) = 1 = f(2)$, but $1 \neq 2$.

YES, ONTO because for any $n \in T$, we have $n + 1 \in S$, and since $n + 1 > 1$, we have $f(n + 1) = n$.

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2. Saracino, Section 7, Problem 7.4: Let $S = T = \{\text{polynomials with real coefficients}\}$, and define a function $D : S \to T$ mapping each polynomial f to its derivative f'. Is this function one-to-one? Is it onto?

Answer/Proof. NO, NOT ONE-TO-ONE; BUT YES, ONTO

Not 1-1: Let $f(x) = 1$ and $g(x) = 2$ (both constant functions). Then $f' = g'$ is the zero function, even though $f \neq g$.

Onto: Given $f \in T$, write $f(x) = \sum_{i=0}^{n} a_i x^i$ with $n \geq 0$ and $a_i \in \mathbb{R}$. Define $g(x) = \sum_{i=0}^{n} \frac{a_i}{i+1} x^i$. Then $\frac{a_i}{i+1} \in \mathbb{R}$ for each $i = 0, \ldots n$ because $i+1 \neq 0$, and so $g \in S$. And by differentiation rules, $D(g) = g' = f$.
QED

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Note: This is just a formal way of saying that derivatives of different functions can end up the same (if they differ by a constant), but that every polynomial has an antiderivative (which is also a polynomial).

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3. Saracino, Section 7, Problem 7.8: Let G be a group, and let $f(x) = x^{-1}$ for all $x \in G$. Is f a function from G to G? If so, is it one-to-one? Is it onto?

Answer/Proof. YES, ONE-TO-ONE AND ONTO

Function: Given $x \in G$, there does indeed exist a unique inverse $x^{-1} \in G$.

1-1: Given $x, y \in G$ with $f(x) = f(y)$, we have $x^{-1} = y^{-1}$. Taking inverse of both sides, we have $x = y$.

Onto: Given $y \in G$, let $x = y^{-1} \in G$. Then $f(x) = (y^{-1})^{-1} = y$ QED

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4. Saracino, Section 7, Problem 7.10(a): Prove that $f : S \to T$ is one-to-one if and only if there exists a function $g : T \to S$ such that $g \circ f = \mathrm{id}_S$. [Note from RLB: you may assume here that $S \neq \varnothing$.]

Proof. ($\Rightarrow$): Pick $a \in S$ [which we can do since we have assumed $S \neq \varnothing$]. Define $g : T \to S$ by

$$g(t) = \begin{cases} \text{the unique } s \in S \text{ such that } f(s) = t & \text{if } t \in f(S), \\ a & \text{otherwise.} \end{cases}$$

Then g is indeed a function from T to S. After all, if $t \in f(S)$, then there *is* some $s \in S$ such that $f(s) = t$; and since f is one-to-one, this s is unique.
To show that $g \circ f = \mathrm{id}_S$, first note that both of these functions are maps from S to S, so they already share the same domain and the same target set. Given $s \in S$, we have

$$g \circ f(s) = g(f(s)) = s = \mathrm{id}_S(s),$$

where the second equality is by our definition of g, since $f(s) \in f(S)$. Thus $g \circ f = \mathrm{id}_S$.
($\Leftarrow$): Given $s_1, s_2 \in S$ with $f(s_1) = f(s_2)$, we have

$$s_1 = \mathrm{id}_S(s_1) = g \circ f(s_1) = g(f(s_1)) = g(f(s_2)) = g \circ f(s_2) = \mathrm{id}_S(s_2) = s_2.$$ QED

Note: The forward implication is technically false if $S = \varnothing$ but $T \neq \varnothing$, because in that case there are **no** functions from T to S. That's why I said you can assume $S \neq \varnothing$.

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5. Saracino, Section 7, Problem 7.12(a): Let $f : S \to T$. For any subset $A \subseteq S$, define $f(A) = \{f(s) \mid s \in A\}$, which is a subset of T. If $A, B \subseteq S$ are subsets, prove that $f(A \cup B) = f(A) \cup f(B)$.

Proof. ($\subseteq$): Given $y \in f(A \cup B)$, there exists $x \in A \cup B$ such that $f(x) = y$.
If $x \in A$, then $y = f(x) \in f(A) \subseteq f(A) \cup f(B)$, as desired.
Otherwise, we must have $x \in B$, so that $y = f(x) \in f(b) \subseteq f(A) \cup f(B)$. QED

($\supseteq$): Given $y \in f(A) \cup f(B)$, we consider two cases.
First, if $y \in f(A)$, then there exists $x \in A$ such that $y = f(x)$. Hence $x \in A \cup B$, so $y = f(x) \in f(A \cup B)$.
Second, the only other possibility is that $y \in f(B)$. Then there exists $x \in B$ such that $y = f(x)$. Hence $x \in A \cup B$, so $y = f(x) \in f(A \cup B)$. QED

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6. Saracino, Section 7, Problem 7.12(b): With notation as in the previous problem, prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. Construct an example where the inclusion is proper, i.e., where $f(A \cap B) \subsetneq f(A) \cap f(B)$.

Proof. Given $y \in f(A \cap B)$, there exists $x \in A \cap B$ such that $f(x) = y$. Thus, $x \in A$, so that $y = f(x) \in f(A)$; and in addition, $x \in B$, so that $y = f(x) \in f(B)$. Hence, $y \in f(A) \cap f(B)$ QED

Counterexample: There are many ways to do this, but we'll do a simple example with small sets. Let $S = \{1, 2\}$ and $T = \{3\}$, and define $f : S \to T$ by $f(1) = f(2) = 3$.
Let $A = \{1\} \subseteq S$ and $B = \{2\} \subseteq S$. Then $f(A) = f(B) = \{3\}$. However, $A \cap B = \varnothing$, so $f(A \cap B) = \varnothing \subsetneq \{3\} = f(A) \cap f(B)$, as desired.

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7. Saracino, Section 8, Problem 8.1: Compute the following multiplications in S_6:

(a): $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 1 & 4 & 2 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 3 & 2 & 1 & 6 \end{pmatrix}$ (b): $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 4 & 1 & 6 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

(c): $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 4 & 1 & 2 \end{pmatrix}$

Solutions. Direct computation gives:

(a): $\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 6 & 3 & 5 \end{pmatrix}}$ Or in cycle notation, that's $(1, 2, 4, 6, 5, 3)$.

(b): $\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 3 & 2 & 5 \end{pmatrix}}$ Or in cycle notation, that's $(1, 6, 5, 2)(3, 4)$.

(c): $\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix}}$ Or in cycle notation, that's $(1, 3, 6, 4)(2, 5)$.