

Solutions to Homework #7

1. Saracino, Section 5, Problem 5.23: Let G be a group, and let $g \in G$. Define the **centralizer** of g in G to be the subset $Z(g) = \{x \in G \mid xg = gx\}$. Prove that $Z(g)$ is a subgroup of G .

Proof. [Clearly $Z(g) \subseteq G$.]

Nonempty: We have $eg = g = ge$, and hence $e \in Z(g)$.

Closure: Given $x, y \in Z(g)$. [We'll show that $xy \in Z(g)$.] We have

$$(xy)g = x(yg) = x(gy) = (xg)y = (gx)y = g(xy).$$

Thus, $xy \in Z(g)$.

Inverses: Given $x \in Z(g)$. [We'll show that $x^{-1} \in Z(g)$.] We have

$$x^{-1}g = x^{-1}gxx^{-1} = x^{-1}xgx^{-1} = gx^{-1}.$$

Thus, $x^{-1} \in Z(g)$. QED

2. Saracino, Section 5, Problem 5.25: Let G be a group and let $a \in G$. Let H be a subgroup of G , and define $aHa^{-1} = \{aha^{-1} \mid h \in H\}$, i.e., the set of all elements of G that are of the form aha^{-1} with $h \in H$. Prove that aHa^{-1} is a subgroup of G .

Proof. [Clearly $aHa^{-1} \subseteq G$.]

(Nonempty): Since H is a subgroup, it is nonempty, so there is some $h \in H$. Then $aha^{-1} \in aHa^{-1}$.

(Closure): Given $x, y \in aHa^{-1}$, there exist $s, t \in H$ such that $x = asa^{-1}$ and $y = ata^{-1}$. Then

$$xy = (asa^{-1})(ata^{-1}) = a(st)a^{-1} \in aHa^{-1},$$

because $st \in H$, since $s, t \in H$ and H is a subgroup.

(Inverses): Given $x \in aHa^{-1}$, there exists $h \in H$ such that $x = aha^{-1}$. Then

$$x^{-1} = (aha^{-1})^{-1} = ah^{-1}a^{-1} \in aHa^{-1},$$

because $h^{-1} \in H$, since $h \in H$ and H is a subgroup. QED

3. Saracino, Section 6, Problem 6.1(c,d): Calculate the order of the element:

c. $(8, 6, 4)$ in $C_{18} \times C_9 \times C_8$

d. $(8, 6, 4)$ in $C_9 \times C_{17} \times C_{10}$

Solution. (c): By Theorem 4.4(iii), the order of 8 in C_{18} is $o(8) = 18/(18, 8) = 18/2 = 9$, the order of 6 in C_9 is $o(6) = 9/(9, 6) = 9/3 = 3$, and the order of 4 in C_8 is $o(4) = 8/(8, 4) = 8/4 = 2$.

By Theorem 6.1(i), we have $o((8, 6, 4)) = \text{lcm}(9, 3, 2) = \boxed{18}$

(d): By Theorem 4.4(iii), the order of 8 in C_9 is $o(8) = 9/(9, 8) = 9/1 = 9$, the order of 6 in C_{17} is $o(6) = 17/(17, 6) = 17/1 = 17$, and the order of 4 in C_{10} is $o(4) = 10/(10, 4) = 10/2 = 5$.

By Theorem 6.1(i), we have $o((8, 6, 4)) = \text{lcm}(9, 17, 5) = \boxed{765}$

4. Saracino, Section 6, Problem 6.2(a,b,d): Which of the following groups are cyclic? [And why?]

a. $C_{12} \times C_9$

b. $C_{10} \times C_{85}$

d. $C_{22} \times C_{21} \times C_{65}$

Solution. (a): We have $(12, 9) = 3 \neq 1$, so the orders of the two cyclic groups in this product are *not* relatively prime. So by Theorem 6.1(ii), the group $C_{12} \times C_9$ is not cyclic

(b): We have $(10, 85) = 5 \neq 1$, so the orders of the two cyclic groups in this product are *not* relatively prime. So by Theorem 6.1(ii), the group $C_{10} \times C_{85}$ is not cyclic

(c): We have $(22, 21) = 1$ [because $22 = 2 \cdot 11$ and $21 = 3 \cdot 7$], and $(22, 65) = 1$ [because $22 = 2 \cdot 11$ and $65 = 5 \cdot 13$], and $(21, 65) = 1$ [because $21 = 3 \cdot 7$ and $65 = 5 \cdot 13$]. That is, any two of the orders of the three cyclic groups in this product are relatively prime. So by Theorem 6.1(ii), the group $C_{22} \times C_{21} \times C_{65}$ is cyclic

5. Saracino, Section 6, Problem 6.6: Prove that $G_1 \times G_2 \times \cdots \times G_n$ is abelian if and only if each G_i is abelian.

Proof. Write $G = G_1 \times \cdots \times G_n$

(\Rightarrow): Given $i \in \{1, \dots, n\}$ and $g_i, h_i \in G_i$, the fact that G is abelian gives

$$\begin{aligned}(e_1, \dots, e_{i-1}, g_i h_i, e_{i+1}, \dots, e_n) &= (e_1, \dots, e_{i-1}, g_i, e_{i+1}, \dots, e_n)(e_1, \dots, e_{i-1}, h_i, e_{i+1}, \dots, e_n) \\ &= (e_1, \dots, e_{i-1}, h_i, e_{i+1}, \dots, e_n)(e_1, \dots, e_{i-1}, g_i, e_{i+1}, \dots, e_n) = (e_1, \dots, e_{i-1}, h_i g_i, e_{i+1}, \dots, e_n),\end{aligned}$$

where e_j denotes the identity element of G_j . Hence, comparing the i -th coordinates, we have $g_i h_i = h_i g_i$.

(\Leftarrow): Given $(g_1, \dots, g_n), (h_1, \dots, h_n) \in G$, we have

$$(g_1, \dots, g_n)(h_1, \dots, h_n) = (g_1 h_1, \dots, g_n h_n) = (h_1 g_1, \dots, h_n g_n) = (h_1, \dots, h_n)(g_1, \dots, g_n),$$

as desired.

QED

[**Note:** On the forward implication (\Rightarrow), we had to actually **pick** elements of each of the other groups G_j for $j \neq i$, because all we were given is that any two **full n -tuples** commute with one another. We can't just write down a partial n -tuple. We also can't say anything like, "pick any elements $g_1 \in G_1, g_2 \in G_2, \dots$," because when choosing elements of sets, we always need to be sure we're **not** inadvertently making the illegal move of selecting an element from the empty set. So really, the only thing we can do here is choose the one element from each other G_j that we know for sure is actually in there: the identity element e_j .]

6. Saracino, Section 6, Problem 6.8: Construct a group of order 81 with the property that every element except the identity has order 3.

Solution. Let $G = C_3 \times C_3 \times C_3 \times C_3$. Then G is a group of order $|G| = 3 \cdot 3 \cdot 3 \cdot 3 = 81$.

Given any $g \in G \setminus \{e\}$, we have $g = (a_1, a_2, a_3, a_4)$ where $a_i \in C_3 = \{0, 1, 2\}$ such that a_1, a_2, a_3, a_4 are not all 0. In particular, each a_i has order either 1 or 3, and at least one a_j has order 3.

By Theorem 6.1(i), then, $o(g) = \text{lcm}(o(a_1), o(a_2), o(a_3), o(a_4)) = 3$, because $o(a_i) | 3$, and $o(a_j) = 3$. QED