Solutions to Homework #5

1. Saracino, Section 5, Problem 5.1(a,c,d): In each case, determine whether or not H is a subgroup of G.
 (a) $G = (\mathbb{R}, +)$, $H = \mathbb{Q}$
 (c) $G = (\mathbb{Z}, +)$, $H = \mathbb{Z}^+$, i.e., $H = \{n \in \mathbb{Z} \,|\, n > 0\}$
 (d) $G = (\mathbb{Q}^\times, \cdot)$, $H = \mathbb{Q}^+$, i.e., $H = \{x \in \mathbb{Q} \,|\, x > 0\}$

Solution. (a): $\boxed{\text{YES, subgroup}}$ [Clearly $H \subseteq G$.]
Nonempty: $0 \in \mathbb{Q}$.
Closure: Given $a, b \in \mathbb{Q}$, we know $a + b \in \mathbb{Q}$ from grade school.
Inverses: Given $a \in \mathbb{Q}$, we know $-a \in \mathbb{Q}$ from grade school. QED

(c): $G = (\mathbb{Z}, +)$, $H = \mathbb{Z}_{\geq 1}$: $\boxed{\text{NO, not subgroup}}$.
Not closed under inverses: $1 \in H$ but its inverse [in G] is $-1 \notin H$. QED

[**Note**: H *is* nonempty and closed under $+$; it's only inverses that fail. In fact, you can use *any* $n \in H$ to note that $-n \notin H$.]

(d): $G = (\mathbb{Q}^\times, \cdot)$, $H = \mathbb{Q}_{>0}$: $\boxed{\text{YES, subgroup}}$ [Clearly $H \subseteq G$.]
Nonempty: $1 \in H$.
Closure: Given $a, b \in H$, then $ab > 0$, and hence $ab \in H$.
Inverses: Given $a \in H$, then $1/a > 0$, and hence $a^{-1} = 1/a \in H$. QED

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2. Saracino, Section 5, Problem 5.1(e,f): In each case, determine whether or not H is a subgroup of G.
 (e) $G = (C_8, \oplus)$, $H = \{0, 2, 4\}$
 (f) $G = (\mathbb{R}^2, +)$, i.e., ordered pairs of real numbers under coordinatewise addition, and
 $H = \{(a, b) \in \mathbb{R}^2 \,|\, b = -a\}$

Solutions. (e): $\boxed{\text{NO, not subgroup}}$
Not closed under $\oplus$: $2, 4 \in H$, but $2 \oplus 4 = 6 \notin H$.

[**Note**: alternatively, not closed under inverses, since $2 \in H$ but $-2 = 6 \notin H$.]

(f): $\boxed{\text{YES, subgroup}}$ [Clearly $H \subseteq G$.]
Nonempty: $(0, 0) \in H$.
Closure: Given $(a, -a), (b, -b) \in H$, we have

$$(a, -a) + (b, -b) = (a + b, -a - b) = (a + b, -(a + b)) \in H.$$

Inverses: Given $(a, -a) \in H$, we have $-(a, -a) = (-a, a) = (-a, -(-a)) \in H$. QED

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3. Saracino, Section 5, Problem 5.2: Let $G = \{\text{functions } f : \mathbb{R} \to \mathbb{R}\}$ be the group of real-valued functions on $\mathbb{R}$, under addition of functions. Let $H = \{f \in G \,|\, f \text{ is differentiable}\}$. Show that H is a subgroup of G.

Proof. [Clearly $H \subseteq G$.]
Nonempty: The zero function $0(x) = 0$ is differentiable [with derivative 0], so $0 \in H$.
Closure: Given $f, g \in H$, we know from calculus that $f + g$ is differentiable, since f and g are differentiable. Thus, $f + g \in H$.
Inverses: Given $f \in H$, we know from calculus that $-f$ is differentiable, since f is differentiable. So $-f \in H$. QED

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4. Saracino, Section 5, Problem 5.3: Let H be the set of elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ of $GL(2, \mathbb{R})$ such that $ad - bc = 1$. Prove that H is a subgroup of $GL(2, \mathbb{R})$.

Proof. [Clearly $H \subseteq G$.]

Nonempty: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in H$.
Closure: Given $A, B \in H$, then $\det(AB) = (\det A)(\det B) = (1)(1) = 1$, where the first equality is by a fact from linear algebra, and the second is by definition of H. Thus, $AB \in H$.
Inverses: Given $A \in H$, then $\det(A^{-1}) = 1/\det(A) = 1/1 = 1$, where the first equality is by a fact from linear algebra, and the second is by definition of H. Thus, $A^{-1} \in H$. QED

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5. Saracino, Section 5, Problem 5.4(a,b):
(a) How many subgroups does $(C_{18}, \oplus)$ have? What are they?
(b) How many subgroups does $(C_{35}, \oplus)$ have? What are they?

Solutions. (a): By Corollary 5.6, the subgroups of the cyclic group C_{18} are precisely the cyclic subgroups $\langle d \rangle$ as d ranges through the positive divisors of 18.
The positive integers dividing 18 are $m = 1, 2, 3, 6, 9, 18$ (but 18 is 0 in C_{18}), so the $\boxed{\text{6 distinct subgroups}}$ of C_{18} are $\boxed{\langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 6 \rangle, \langle 9 \rangle, \langle 0 \rangle}$
(b): By Corollary 5.6, the subgroups of the cyclic group C_{35} are precisely the cyclic subgroups $\langle d \rangle$ as d ranges through the positive divisors of 35.
The positive integers dividing 35 are $m = 1, 5, 7, 35$ (but 35 is 0 in C_{35}), so the $\boxed{\text{4 distinct subgroups}}$ of C_{35} are $\boxed{\langle 1 \rangle, \langle 5 \rangle, \langle 7 \rangle, \langle 0 \rangle}$

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6. Saracino, Section 5, Problem 5.10: Prove that every subgroup of an abelian group is abelian.

Proof. Let G be an abelian group, and let $H \subseteq G$ be a subgroup.
Given $x, y \in H$, we have $x, y \in G$. Since G is abelian, $xy = yx$. QED