Math 350, Spring 2026 Professor Rob Benedetto

Solutions to Homework #2

1. Saracino, Section 2, Problem 2.4(a,b).

Answers. Doing addition mod 4 and mod 5, respectively, we get:

(a): C_4:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(b): C_5:

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

2. Saracino, Section 2, Problem 2.5.

Proof. $\boxed{\text{NO}}$ S is not a group. We prove this by contradiction. If S were a group, then the identity must be $e = a$, because the table shows that for each $x \in S$,we have $a * x = x = x * a$. But then c does not have an inverse, because for every $x \in S$, we have $c * x = c \neq e$.

Note 1: We'll soon see that for a group, every element of the group appears exactly once in every row of the multiplication table, and exactly once in every column. So the repeats in some of the rows and columns are an immediate indication that S is not a group.

Note 2: Incidentally, this binary operation **is** associative. (And as we saw above, it has an identity, so it's only the failure of the inverse axiom that prevents it from being a group.) Checking that it's associative from the group table alone would be hard; you've have to check all $3^3 = 27$ choices of what $x, y, z \in \{a, b, c\}$ could be when testing the equation $(x * y) * z \overset{?}{=} x * (y * z)$. Instead, in this case there is actually a simple description of the operation: if we think of a, b, c as numbers with $a < b < c$, the operation here is $x * y = \max\{x, y\}$, i.e., just output whichever of x, y comes later in alphabetical order. Once you realize that, it's easy to see that for any of the 27 choices of x, y, z, we have

$$(x * y) * z = \max\big\{ \max\{x, y\}, z\big\} = \max\{x, y, z\} = \max\big\{x, \max\{y, z\}\big\} = x * (y * z).$$

But this is all just a side comment, because you didn't need to show it's associative; you just needed to find one thing that broke in the group axioms, not show all the things that didn't break. The three-line proof I gave above does the trick.

3. Saracino, Section 2, Problem 2.8: Let G be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ which have the property that $f(x) \neq 0$ for all $x \in \mathbb{R}$. Define the product of $f, g \in G$ by

$$(f \times g)(x) = f(x)g(x) \quad \text{for all } x \in \mathbb{R}.$$

Is $(G, \times)$ a group? Prove or disprove.

Answer/Proof. $\boxed{\text{YES}}$ G is a group:

Bin Op: Given $f, g \in G$, then for any $x \in \mathbb{R}$, we have $(f \times g)(x) = f(x)g(x) \in \mathbb{R}$, and moreover $(f \times g)(x) \neq 0$ since $f(x), g(x) \neq 0$. Thus $f \times g \in G$.

Assoc: Given $f, g, h \in G$, then for any $x \in \mathbb{R}$, we have

$$\begin{aligned}\big((f \times g) \times h\big)(x) &= (f \times g)(x)h(x) = \big(f(x)g(x)\big)h(x) = f(x)\big(g(x)h(x)\big) \\ &= f(x)(g \times h)(x) = \big(f \times (g \times h)\big)(x)\end{aligned}$$

Thus, $(f \times g) \times h = f \times (g \times h)$.

Identity. Let $e(x) = 1$ for all $x \in \mathbb{R}$. Then $e \in G$ since e is real-valued and never 0. For any $f \in G$ and any $x \in \mathbb{R}$, we have

$$(f \times e)(x) = f(x)e(x) = f(x) \cdot 1 = f(x) \quad \text{and} \quad (e \times f)(x) = e(x)f(x) = 1 \cdot f(x) = f(x).$$

Thus, $f \times e = f = e \times f$.

Inverses. Given $f \in G$, define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 1/f(x)$, which is defined and real for each $x \in \mathbb{R}$ because $f(x)$ is nonzero and real. We also have $g(x) \neq 0$ for each $x \in \mathbb{R}$. Thus, $g \in G$. Moreover, for any $x \in \mathbb{R}$, we have

$$(f \times g)(x) = f(x) \cdot \frac{1}{f(x)} = 1 = e(x) \quad \text{and} \quad (g \times f)(x) = \frac{1}{f(x)} \cdot f(x) = 1 = e(x).$$

Thus, $f \times g = e = g \times f$. QED

---

4. Saracino, Section 2, Problem 2.11: Let G be the set of all 2×2 matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b \in \mathbb{R}$ and $a, b \in \mathbb{R} \smallsetminus \{0\}$ are nonzero real numbers. Prove that G forms a group under matrix multiplication.

Proof. **Bin Op**: Given $A, B \in G$, we may write $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $B = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$ are nonzero. Then $AB = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$. Since $ac, bd \in \mathbb{R}$ with $ac, bd \neq 0$, we have $AB \in G$.

Assoc: Given $A, B, C \in G$, we have $(AB)C = A(BC)$ because matrix multiplication is associative (from linear algebra).

Identity: Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, which is in G because $1 \in \mathbb{R}$ is nonzero. For any $A \in G$, we have $IA = A = AI$ because I is the identity matrix and we know this fact from linear algebra.

Inverses: Given $A \in G$, we may write $A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with $a, b \in \mathbb{R}$ nonzero. Then $1/a, 1/b$ are defined and are also nonzero real numbers. So we may define $B = \begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$, and B is an element of G. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = BA,$$

so $AB = I = BA$, as desired. QED

---

5. Saracino, Section 3, Problem 3.3: Find elements $A, B, C \in GL(2, \mathbb{R})$ such that $AB = BC$ but $A \neq C$.

Solution Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.
Then $\det A = 2 \neq 0$, $\det B = 1 \neq 0$, and $\det C = 2 \neq 0$, so A, B, C are 2×2 invertible matrices with real entries, i.e., $A, B, C \in GL(2, \mathbb{R})$.
We compute $AB = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ and $BC = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = AB$.
But clearly $A \neq C$. QED

Note: There are *many* correct solutions to this problem. Since $B \in GL(2, \mathbb{R})$ is invertible, the equation $AB = BC$ is equivalent to $C = B^{-1}AB$, i.e., C is conjugate to A, a situation you would have seen happen a lot in linear algebra, especially when you learned about diagonalizing matrices.

---

6. Saracino, Section 3, Problem 3.4: Let $(G, *)$ be a group, and let $g \in G$. Suppose that there is (at least) one element $x \in G$ such that $x * g = x$. Prove that $g = e$.

Proof. Since G is a group, there is an inverse $x^{-1} \in G$ of x. Then we have

$$g = e * g = (x^{-1} * x) * g = x^{-1} * (x * g) = x^{-1} * x = e$$

as desired, where the first equality is by the identity axiom, the second and fifth are by the inverse axiom, the third is by the associativity axiom, and the fourth is by hypothesis. QED