Solutions to Homework #15

1. Saracino, Section 12, Problem 12.4(c,f):
In each case, determine whether or not the two groups are isomorphic.

Solutions. (c): $\mathbb{R}^\times$ and $\mathbb{R}$ are $\boxed{\text{Not isomorphic}}$

Note that $-1 \in \mathbb{R}^\times$ has order 2, since $-1 \neq 1$, but $(-1)^2 = 1$. However, no element of $\mathbb{R}$ has order 2, since if $x \in \mathbb{R}$ satisfies $2x = 0$, then $x = 0$, meaning that $o(x) = 1 \neq 2$. By Theorem 12.5(iv), then, there cannot be an isomorphism from $\mathbb{R}^\times$ to $\mathbb{R}$. QED

(f): $\mathbb{R}^\times$ and $\mathbb{R}_{>0} \times C_2$ are $\boxed{\text{isomorphic}}$

Define $\varphi : \mathbb{R}_{>0} \times C_2 \to \mathbb{R}^\times$ by

$$\varphi(x, m) = (-1)^m x \quad \text{for each } x \in \mathbb{R} \text{ and } m \in C_2 = \{0, 1\}.$$

Then φ is defined because for any such x and m, we have $(-1)^m x \in \mathbb{R}$ and $(-1)^m x \neq 0$, since $x \neq 0$.

(Homom): Given $(x, m), (y, n) \in \mathbb{R}_{>0} \times C_2$, we have

$$\varphi((x, m) * (y, n)) = \varphi(xy, m + n) = (-1)^{m+n} xy = [(-1)^m x][(-1)^n y] = \varphi(x, m)\varphi(y, n).$$

(1-1): Given $(x, m), (y, n) \in \mathbb{R}_{>0} \times C_2$ with $\varphi(x, m) = \varphi(y, n)$, we have

$$x = |(-1)^m x| = |\varphi(x, m)| = |\varphi(y, n)| = |(-1)^n y| = y.$$

Moreover, if $\varphi(x, m) > 0$, then $(-1)^m, (-1)^n > 0$, so that $m, n \neq 1$, and hence $m = n = 0$. Otherwise, we have $\varphi(x, m) < 0$, and therefore $(-1)^m, (-1)^n < 0$, so that $m, n \neq 0$, and hence $m = n = 1$.

(Onto): Given $t \in \mathbb{R}^\times$, if $t > 0$, then $(t, 0) \in \mathbb{R}_{>0} \times C_2$, and $\varphi(t, 0) = (-1)^0 t = t$. Otherwise, we have $t < 0$, whence $(-t, 1) \in \mathbb{R}_{>0} \times C_2$, and $\varphi(-t, 1) = (-1)^1(-t) = t$.

Thus, φ is an isomorphism, confirming that $\mathbb{R}^\times \cong \mathbb{R}_{>0} \times C_2$. QED

[**Alternative Method for (f)**: Define $\psi : \mathbb{R}^\times \to \mathbb{R}_{>0} \times C_2$ by $\psi(t) = \begin{cases} (t, 0) & \text{if } t > 0, \\ (-t, 1) & \text{if } t < 0. \end{cases}$

Then prove that ψ is a homomorphism, is one-to-one, and is onto. I'll skip those details here, but the proof is a little stringier because multiple cases are required, especially to prove ψ is a homomorphism.]

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2. Saracino, Section 12, Problem 12.13: Let $\varphi : G \to H$ be a homomorphism.

(a) If H is abelian and φ is one-to-one, prove that G is abelian.
(b) If G is abelian and φ is onto, prove that H is abelian.
(c) If φ is an isomorphism, prove that G is abelian if and only if H is abelian.

Proof. (a): Given $x, y \in G$, then

$$\varphi(xy) = \varphi(x)\varphi(y) = \varphi(y)\varphi(x) = \varphi(yx),$$

where the second equality is because H is abelian. Since φ is 1-1, then, we have $xy = yx$. QED

(b): Given $a, b \in H$, there exist $x, y \in G$ such that $\varphi(x) = a$ and $\varphi(y) = b$. Thus,

$$ab = \varphi(x)\varphi(y) = \varphi(xy) = \varphi(yx) = \varphi(y)\varphi(x) = ba,$$

where the third equality is because G is abelian. QED

(c) [If φ isom, then G abelian iff H abelian.] ($\Rightarrow$): Since φ is onto, we are done by part (b).
($\Leftarrow$): Since φ is one-to-one, we are done by part (a). QED

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3. Saracino, Section 12, Problem 12.15: Let $\varphi : G \to H$ be an onto homomorphism. If G is cyclic, prove that H is also cyclic.

Proof. Let $a \in G$ be a generator for G. It suffices to show that $\varphi(a)$ is a generator for H.
Given $h \in H$, there is some $g \in G$ such that $\varphi(g) = h$, since φ is onto. Because $G = \langle a \rangle$, there is some $n \in \mathbb{Z}$ such that $g = a^n$. Thus, $h = \varphi(g) = \varphi(a^n) = \varphi(a)^n$. QED

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4. Saracino, Section 12, Problem 12.21: Let G be the group $\mathbb{C}^\times$ of nonzero complex numbers under multiplication, and let

$$H = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| \, a, b \in \mathbb{R} \text{ are not both } 0 \right\}.$$

You may take my word for it that H is a subgroup of $GL(2, \mathbb{R})$. Prove that $G \cong H$.

Proof. Define $\varphi : G \to H$ by $\varphi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Note that φ is defined because any $a + bi \in \mathbb{C}^\times$ has $a, b \in \mathbb{R}$ with a, b not both zero.

(Homom): Given $a + bi, c + di \in \mathbb{C}^\times$, we have

$$\varphi((a+bi)(c+di)) = \varphi((ac-bd)+(ad+bc)i) = \begin{bmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{bmatrix} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$$
$$= \varphi(a+bi)\varphi(c+di).$$

(1-1): Given $a + bi, c + di \in \mathbb{C}^\times$ with $\varphi(a + bi) = \varphi(c + di)$, we have $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$, so that $a = c$ and $b = d$, and hence $a + bi = c + di$.

(Onto): Given $\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \in H$, we have $a, b \in \mathbb{R}$ not both zero, and hence $a + bi \in \mathbb{C}^\times$ with

$\varphi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Thus, φ is an isomorphism, so $G \cong H$. QED

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5. Saracino, Section 12, Problem 12.22: Let G be a group, and let $g \in G$. Define $\varphi : G \to G$ by $\varphi(x) = gxg^{-1}$. Prove that φ is an automorphism of G.

Proof. **(Homom)**: Given $x, y \in G$, we have

$$\varphi(xy) = g(xy)g^{-1} = (gxg^{-1})(gyg^{-1}) = \varphi(x)\varphi(y).$$

(1-1): Given $x, y \in G$ such that $\varphi(x) = \varphi(y)$, then $gxg^{-1} = gyg^{-1}$, so by the cancellation laws, we have $x = y$.

(Onto): Given $y \in G$, let $x = g^{-1}yg \in G$. Then

$$\varphi(x) = g(g^{-1}yg)g^{-1} = eye = y$$ QED

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6. Saracino, Section 13, Problem 13.1: Let $\varphi : C_8 \to C_4$ be given by $\varphi(x) =$ remainder of $x \pmod 4$. Find $\ker(\varphi)$. Also, to which familiar group is $C_8/\ker\varphi$ isomorphic?

Solution. For any $m \in C_8$, we have $\varphi(m) = 0 \in C_4$ if and only if $m \equiv 0 \pmod 4$, i.e., if and only if $m = 0, 4$. Thus, $\boxed{\ker(\varphi) = \{0, 4\}}$

Noting that φ is onto (because $\varphi(m) = m$ for $m = 0, 1, 2, 3$) and a homomorphism, the Fundamental Theorem (Theorem 13.2) says that $C_8/\ker\varphi$ is isomorphic to $\boxed{C_4}$

[Alternative proof of second statement: We have $|C_8/\ker\varphi| = |C_8|/|\ker\varphi| = 8/2 = 4$, and any quotient of a cyclic group is cyclic (Exercise 11.18), so $C_8/\ker\varphi$ is cyclic of order 4 and hence isomorphic to C_4 by Theorem 12.2.]