

Solutions to Homework #12

1. Saracino, Section 10, Problem 10.15: Let G be a finite group, and let H be a subgroup of G . Let K be a subgroup of H . Prove that $[G : K] = [G : H][H : K]$.

Proof. Let $n = |G|$, $m = |H|$, and $k = |K|$, all of which are finite, because G is a finite group. Also noted that K is a subgroup of G as well, because it is a subset of G that is a group.

By Lagrange's Theorem, we have $[G : K] = \frac{n}{k}$ and $[G : H] = \frac{n}{m}$ and $[H : K] = \frac{m}{k}$. Thus,

$$[G : K] = \frac{n}{k} = \frac{n}{m} \cdot \frac{m}{k} = [G : H][H : K]. \quad \text{QED}$$

2. Saracino, Section 10, Problem 10.23(a): Let S = the set of even integers, and let T = the set of odd integers.

Prove that S and T have the same cardinality.

Proof. Define $f : S \rightarrow T$ by $f(n) = n + 1$. This is a function from S to T , because for any even integer n , we know that $n + 1$ is odd.

One-to-one: Given $m, n \in S$ such that $f(m) = f(n)$, we have $m + 1 = n + 1$, so $m = n$.

Onto: Given $k \in T$, let $n = k - 1$. Since k is an odd integer, we know that n is even, and hence $n \in S$. Moreover, $f(n) = (k - 1) + 1 = k$. QED

3. Saracino, Section 10, Problem 10.25: Find the conjugacy classes in Q_8 , and write down the class equation for Q_8 .

Solution. Write $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. From our previous work with Q_8 , we know that $Z(Q_8) = \{\pm 1\}$. Thus, $\{1\}$ and $\{-1\}$ are both (one-element) conjugacy classes.

We claim that the conjugacy class of i is $\{\pm i\}$. Certainly i is conjugate to itself, and we compute

$$jij^{-1} = -k(-j) = kj = -i.$$

Thus, the conjugacy class $[i]_c$ of i contains $\{\pm i\}$. Meanwhile, the centralizer $Z(i)$ contains $\langle i \rangle = \{\pm 1, \pm i\}$ and hence has at least four elements. [In fact, $Z(i) = \langle i \rangle$.] Hence, by Lemma 10.8, the number of elements in $[i]_c$ is $[Q_8 : Z(i)] = |Q_8|/|Z(i)| \leq 8/4 = 2$ elements. Since we already know of the two elements $\pm i$, we have proven our claim that $[i]_c = \{\pm i\}$.

Similarly, $[j]_c = \{\pm j\}$ and $[k]_c = \{\pm k\}$ are conjugacy classes. So there are five conjugacy classes: $\{1\}$, $\{-1\}$, $\{\pm i\}$, $\{\pm j\}$, $\{\pm k\}$.

Thus, the class equation says $|Q_8| = |Z(Q_8)| + |[i]_c| + |[j]_c| + |[k]_c|$, i.e., $8 = 2 + 2 + 2 + 2$.

4. Saracino, Section 10, Problem 10.28:

Let p be a prime number, and let n be a positive integer. Let G be a group with $|G| = p^n$. Use the class equation to prove that $|Z(G)|$ is divisible by p .

Proof. Let a_1, \dots, a_k be representatives for the distinct conjugacy classes with at least two elements. For each $i = 1, \dots, k$, Lagrange's Theorem gives

$$p^n = |G| = |Z(a_i)||[G : Z(a_i)]|,$$

and hence $[G : Z(a_i)]$ is a divisor of p^n . That is, $[G : Z(a_i)]$ is one of $1, p, p^2, \dots, p^n$.

On the other hand, since $[G : Z(a_i)]$ is the size of a_i 's conjugacy class, we have $[G : Z(a_i)] \geq 2$ by our choice of a_i . Thus, $[G : Z(a_i)]$ is one of p, p^2, \dots, p^n . In particular, $p|[G : Z(a_i)]$.

By the class equation, we have

$$|Z(G)| = |G| - [G : Z(a_1)] - \dots - [G : Z(a_k)] = p^n - [G : Z(a_1)] - \dots - [G : Z(a_k)].$$

By the above, every term on the right side of this equation is an integer divisible by p . Thus, $|Z(G)|$ is divisible by p . QED

5. Saracino, Section 10, Problem 10.29:

Let p be a prime number and let G be a group such that $|G| = p^2$. Prove that G is abelian.

Proof. By Lagrange's Theorem, the center $Z(G)$ has order either 1, p , or p^2 . By Exercise 10.28, $|Z(G)| \neq 1$.

Suppose (towards contradiction) that $|Z(G)| = p$. Then since $|G| = p^2 > p$, there is some element $a \in G$ not in $Z(G)$. Note that the centralizer subgroup $Z(a)$ contains $Z(G)$ [since everything in the center commutes with everything, and hence commutes with a] as well as a . [since a commutes with itself]. Since $a \notin Z(G)$, then we have found $p + 1$ distinct elements of $Z(a)$. Thus, $|Z(a)| \geq p + 1$. Again by Lagrange, meanwhile, $Z(a)$ must have either 1, p , or p^2 elements; thus, $|Z(a)| = p^2$. Thus, $Z(a) = G$, which means that a commutes with everything in G , and hence $a \in Z(G)$, a contradiction.

We have eliminated the possibilities that $|Z(G)|$ is 1 or p . Hence, $|Z(G)| = p^2$, and therefore $Z(G) = G$. That is, every element of G commutes with every element of G ; in other words, G is abelian. QED

6. Saracino, Section 11, Problem 11.1:

Recall that $SL(2, \mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}) \mid ad - bc = 1 \right\}$.

We have already seen that $SL(2, \mathbb{R})$ is a subgroup of $GL(2, \mathbb{R})$. Prove that $SL(2, \mathbb{R}) \triangleleft GL(2, \mathbb{R})$.

Proof. We already know $SL(2, \mathbb{R})$ is a subgroup of $GL(2, \mathbb{R})$, from some earlier homework problem.

Given $A \in SL(2, \mathbb{R})$ and $B \in GL(2, \mathbb{R})$, we have $BAB^{-1} \in GL(2, \mathbb{R})$, and

$$\det(BAB^{-1}) = \det(B) \det(A) [\det(B)]^{-1} = \det(B) \cdot 1 \cdot \frac{1}{\det(B)} = 1,$$

so $BAB^{-1} \in SL(2, \mathbb{R})$.

QED

7. Saracino, Section 11, Problem 11.2:

Let H be the subgroup of $G = GL(2, \mathbb{R})$ consisting of all matrices $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ such that $ad \neq 0$. Is H a normal subgroup of G ? Why or why not?

Answer/Proof. NO, H is not normal in $GL(2, \mathbb{R})$ To see this, we just need to find two matrices $A \in H$ and $B \in GL(2, \mathbb{R})$ such that $BAB^{-1} \notin H$.

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in H$, and let $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in GL(2, \mathbb{R})$. Then $B^{-1} = B$, and hence

$$BAB^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \notin H \quad \text{QED}$$

Note: There are many choices of A, B that will work here. Don't forget that they both need to be invertible, though!