

Solutions to Homework #11

1. Saracino, Section 9, Problem 9.16(a,b):

Let H be a subgroup of a group G , and define a relation $H \equiv$ on G by $x_H \equiv y$ iff $x^{-1}y \in H$.

a. Show that $H \equiv$ is an equivalence relation on G .

b. Show that the equivalence classes under $H \equiv$ are precisely the left cosets of H in G .

Proof. (a): (**Ref**): Given $x \in G$, we have $x^{-1}x = e \in H$, so $x_H \equiv x$.

(**Symm**): Given $x, y \in G$ such that $x_H \equiv y$, we have $y^{-1}x = (x^{-1}y)^{-1} \in H$ since $x^{-1}y \in H$. Thus, $y_H \equiv x$.

(**Tran**): Given $x, y, z \in G$ such that $x_H \equiv y$ and $y_H \equiv z$, we have $x^{-1}z = (x^{-1}y)(y^{-1}z) \in H$, since $x^{-1}y \in H$ and $y^{-1}z \in H$. Thus, $x_H \equiv z$. QED

(b): Given $y \in G$, we must show that $\bar{y} = yH$, where \bar{y} is the equivalence class of y under $H \equiv$.

Given $x \in G$, we have:

$$\begin{aligned} x \in \bar{y} &\iff x_H \equiv y \iff y_H \equiv x \iff y^{-1}x \in H \\ &\iff \exists h \in H \text{ s.t. } y^{-1}x = h \iff \exists h \in H \text{ s.t. } x = yh \iff x \in yH, \end{aligned}$$

as desired. QED

2. Saracino, Section 10, Problem 10.1 (well, part of it, and in my notation):

Let $G = Q_8$. Find $[G : H]$ for $H = \langle -1 \rangle$, and find $[G : K]$ for $K = \langle j \rangle$. [As always, justify your answers.]

Solution. $H = \langle -1 \rangle = \{\pm 1\}$ has $|H| = 2$, so by Lagrange, we have $[G : H] = |G|/|H| = 8/2 = \boxed{4}$

$K = \langle -j \rangle = \{\pm 1, \pm j\}$ has $|K| = 4$, so by Lagrange, we have $[G : K] = |G|/|K| = 8/4 = \boxed{2}$

3. Saracino, Section 10, Problem 10.2(b):

Let $G = C_{54}$. Find $[G : H]$ for $H = \langle 24 \rangle$.

Solution. By Theorem 4.4(iii), $|H| = o(24) = \frac{54}{(24, 54)} = \frac{54}{6} = 9$.

Thus, by Lagrange, we have $[G : H] = |G|/|H| = 54/9 = \boxed{6}$

4. Saracino, Section 10, Problem 10.3(b):

Let $G = C_6 \times C_4$, and let $H = \langle 2 \rangle \times \langle 2 \rangle$. Find $[G : H]$.

Proof. By Theorem 4.4(iii), the order of 2 in C_6 is $6/(2, 6) = 6/2 = 3$, and the order of 2 in C_4 is $4/(2, 4) = 4/2 = 2$.

Thus, by Corollary 4.6, the subgroup $\langle 2 \rangle$ of C_6 has $|\langle 2 \rangle| = 3$, and the subgroup $\langle 2 \rangle$ of C_4 has $|\langle 2 \rangle| = 2$.

Hence, $|H| = 3 \cdot 2 = 6$. Of course, $|G| = 6 \cdot 4 = 24$.

Therefore, by Lagrange, we have $[G : H] = |G|/|H| = 24/6 = \boxed{4}$

5. Saracino, Section 10, Problem 10.5: Let G be a group of order 8 that is not cyclic. For every $a \in G$, prove that $a^4 = e$.

Proof. Given $a \in G$, let $n = o(a)$. By a corollary to Lagrange (Theorem 10.4), we have $n|8$, so n is one of 1, 2, 4, 8.

If $n = 8$, then $|\langle a \rangle| = o(a) = 8$, so that $\langle a \rangle$ must be all of G , and hence a generates G , contradicting the assumption that G is not cyclic. Thus, $n \neq 8$.

So $o(a)$ is one of 1, 2, 4. Therefore, we have $o(a)|4$. Thus, $a^4 = e$. QED

6. Saracino, Section 10, Problem 10.6:

Let G be a group, and let $H, K \subseteq G$ be subgroups. Suppose that $|H| = 12$ and $|K| = 5$. Prove that $H \cap K = \{e\}$.

Proof. From an earlier exercise, $H \cap K$ is a subgroup of both H and K . Let $m = |H \cap K|$.

By Lagrange applied to $H \cap K$ as a subgroup of H , we have $m|12$.

By Lagrange applied to $H \cap K$ as a subgroup of K , we have $m|5$.

Thus, $m \geq 1$ is a common divisor of 12 and 5. Since $(12, 5) = 1$, we have $m = 1$.

Since $e \in H \cap K$, then, we must have $H \cap K = \{e\}$.

QED

7. Saracino, Section 10, Problem 10.8:

Let G be a group of order p^2 , where p is prime. Prove that G must have a subgroup of order p .

Proof. Since $p \geq 2$, we have $|G| \geq 4$, so that $|G \setminus \{e\}| \geq 3$. In particular, there is some $a \in G \setminus \{e\}$.

By a corollary to Lagrange (Theorem 10.4), we have $o(a)|p^2$, so $o(a)$ is either 1, p , or p^2 . Since $a \neq e$, we have $o(a) \neq 1$. Thus, there are only two cases to consider.

Case 1: $o(a) = p$. Then by Corollary 4.6, we have $|\langle a \rangle| = p$, so $\langle a \rangle$ is a subgroup of order p .

Case 2: $o(a) = p^2$. Then by Theorem 4.4, $o(a^p) = \frac{p^2}{(p, p^2)} = \frac{p^2}{p} = p$. Therefore, by Corollary 4.6, we have $|\langle a^p \rangle| = p$, so $\langle a^p \rangle$ is a subgroup of order p .

QED