Solutions to Homework #10

1. Saracino, Section 8, Problem 8.14:
Find $Z(S_3)$ and $Z(D_4)$.

Solution. **(a)**: We claim $\boxed{Z(S_3) = \{e\}}$
Clearly $e \in Z(S_3)$, since it commutes with everything in the group.
We have $(1,2)(1,2,3) = (2,3) \neq (1,3) = (1,2,3)(1,2)$, and hence $(1,2) \notin Z(S_3)$ and $(1,2,3) \notin Z(S_3)$.
We have $(1,3)(1,3,2) = (2,3) \neq (1,2) = (1,3,2)(1,3)$, and hence $(1,3) \notin Z(S_3)$ and $(1,3,2) \notin Z(S_3)$.
Finally, $(1,2)(2,3) = (1,2,3) \neq (1,3,2) = (2,3)(1,2)$, and hence $(1,2) \notin Z(S_3)$. QED

(b): We claim $\boxed{Z(D_4) = \{e, f^2\}}$
Clearly $e \in Z(D_4)$. For any $i \in \{0,1,2,3\}$, we have

$$f^i f^2 = f^{i+2} = f^2 f^i, \quad \text{and} \quad (f^i g)f^2 = f^{i-2}g = f^{i+2}g = f^2(f^i g),$$

confirming that $f^2 \in Z(D_4)$.
Meanwhile, $f^3 g = gf \neq gf^3$, and hence $f^3, g \notin Z(D_4)$.
Also, $f(fg) = f^2 g = fgf^{-1} \neq (fg)f$, and hence $f, fg \notin Z(D_4)$.
Finally, $(f^2 g)(f^3 g) = f^{2-3}g^2 = f^{-1} \neq f = f^{3-2}g^2 = (f^3 g)(f^2 g)$, and hence $f^2 g, f^3 g \notin Z(D_4)$. QED

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2. Saracino, Section 8, Problem 8.15(b,c):
With notation as in the book for the elements of D_n:

(b) Calculate $(f^2 g)(f^3 g)(f^2)$ in D_5. [And show your work.]
(c) Find $Z(D_5)$. [And prove your answer.]

Solution. **(b)**: Using the rules $gf^i = f^{-i}g$ and $f^5 = e$ and $g^2 = e$, we compute:

$$(f^2 g)(f^3 g)(f^2) = f^2(f^{-3}g)(gf^2) = f^2 f^{-3} f^2 = f^1 = \boxed{f}$$

(c): We claim $\boxed{Z(D_5) = \{e\}}$
Clearly $e \in Z(D_5)$, since it commutes with everything in the group.
For each $i = 1, \ldots, 4$, we have $gf^i = f^{-i}g = f^{5-i}g \neq f^i g$, and hence $f^i, g \notin Z(D_5)$.
Still for each $i = 1, \ldots, 4$, we have $g(f^i g) = f^{-i}g^2 = f^{5-i} \neq f^i = (f^i g)g$, and hence $f^i g \notin Z(D_5)$ as well. QED

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3. Saracino, Section 9, Problem 9.1:
Determine which of the following relations R on $\mathbb{Z}$ are equivalence relations (and justify, of course).

(a) $a\,R\,b$ iff $a - b \geq 0$ (b) $a\,R\,b$ iff $|a| = |b|$
(c) $a\,R\,b$ iff $ab \geq 0$ (d) $a\,R\,b$ iff $|a - b| \leq 1$

Solution. (a): $\boxed{\text{NOT equivalence relation}}$ This relation is not symmetric: for example, $4, 5 \in \mathbb{Z}$ with $5\,R\,4$, because $5 - 4 \geq 0$. However, $4 - 5 \not\geq 0$, so $4 \not R\, 5$.

(b): $\boxed{\text{YES, equivalence relation}}$ **(refl)**: Given $a \in \mathbb{Z}$, we have $|a| = |a|$, so $a\,R\,a$.
(symm): Given $a, b \in \mathbb{Z}$ with $a\,R\,b$, we have $|a| = |b|$. Then $|b| = |a|$, so $b\,R\,a$.
(tran): Given $a, b, c \in \mathbb{Z}$ with $a\,R\,b$ and $b\,R\,c$, we have $|a| = |b|$ and $|b| = |c|$, so $|a| = |c|$. Thus, $a\,R\,c$. QED

(c): $\boxed{\text{NOT equivalence relation}}$ This relation is not transitive: for example, $-1, 0, 1 \in \mathbb{Z}$ with $-1\,R\,0$, because $(-1)(0) = 0 \geq 0$, and $0\,R\,1$, because $(0)(1) = 0 \geq 0$. However, $-1 \not R\, 1$, because $(-1)(1) = -1 \not\geq 0$.

(d): $\boxed{\text{NOT equivalence relation}}$ This relation is not transitive: for example, $1, 2, 3 \in \mathbb{Z}$ with $1\,R\,2$, because $|1 - 2| = 1 \leq 1$, and $2\,R\,3$, because $|2 - 3| = 1 \leq 1$. However, $1 \not R\, 3$, because $|1 - 3| = 2 \not\leq 1$

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4. Saracino, Section 9, Problem 9.4:
Let R be an equivalence relation on a set S. Prove that for all $s_1, s_2 \in S$, we have:
$\overline{s_1} = \overline{s_2}$ iff $s_1 \, R \, s_2$.

Proof. Given $s_1, s_2 \in S$:
($\Rightarrow$): By reflexivity, we have $s_1 \, R \, s_1$, and hence $s_1 \in \overline{s_1} = \overline{s_2}$. But the statement $s_1 \in \overline{s_2}$ means precisely that $s_1 \, R \, s_2$. QED ($\Rightarrow$)

($\Leftarrow$):
($\subseteq$): Given $t \in \overline{s_1}$, we have $t \, R \, s_1$ and $s_1 \, R \, s_2$. By transitivity, then, $t \, R \, s_2$. That is, $t \in \overline{s_2}$.

($\supseteq$): Given $t \in \overline{s_2}$, we have $t \, R \, s_2$ and $s_1 \, R \, s_2$. By symmetry, the latter becomes $s_2 \, R \, s_1$. By transitivity, then, $t \, R \, s_1$. That is, $t \in \overline{s_1}$. QED

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5. Saracino, Section 9, Problem 9.5: Find the right cosets of H in $G = Q_8$ for:

a. $H = \langle i \rangle$ (which Saracino denotes $\langle J \rangle$) b. $H = \langle -1 \rangle$ (which Saracino denotes $\langle -I \rangle$)

Solution. (a): We have $H = \{1, i, -1, -i\}$, so $H1 = H$. [And since the elements $i, -1, -i$ appear in the coset, it must also be $Hi = H(-1) = H(-i) = H$.] Arbitrarily choosing another element of H — say the element j — computing hj for each $h \in H$ gives $Hj = \{j, k, -j, -k\}$. [And since the elements $k, -j, -k$ appear in the coset, it must also be $Hj = Hk = H(-j) = H(-k)$.] We have now seen all eight elements of Q_8.
So the (two) cosets of H are $\boxed{H1 = \{1, i, -1, -i\} \text{ and } Hj = \{j, k, -j, -k\}}$

(b): We have $H = \{1, -1\}$, so $H1 = H(-1) = H$.
Multiplying H by i, we have $Hi = \{i, -i\}$, which must also be $H(-i)$.
Multiplying H by j, we have $Hj = \{j, -j\}$, which must also be $H(-j)$.
Multiplying H by k, we have $Hk = \{k, -k\}$, which must also be $H(-k)$.
We have now seen all eight elements of Q_8.
So the (four) cosets of H are $\boxed{H1 = \{1, -1\},\ Hi = \{i, -i\},\ Hj = \{j, -j\}, \text{ and } Hk = \{k, -k\}}$

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6. Section 9, Problem 9.12: Let G be a group. Define a relation R on G by

$$a \, R \, b \quad \text{iff} \quad \exists x \in G \text{ such that } a = xbx^{-1}.$$

Prove that R is an equivalence relation on G.

Proof. (**refl**): Given $a \in G$, we have $a = eae^{-1}$, so $a \, R \, a$ since $e \in G$.
(**symm**): Given $a, b \in G$ such that $a \, R \, b$, we have $a = xbx^{-1}$ for some $x \in G$. Let $y = x^{-1} \in G$. Then $b = x^{-1}ax = yay^{-1}$, and hence $b \, R \, a$.
(**tran**): Given $a, b, c \in G$ such that $a \, R \, b$ and $b \, R \, c$, we have $a = xbx^{-1}$ and $b = ycy^{-1}$ for some $x, y \in G$. Let $z = xy \in G$. Then

$$a = xbx^{-1} = xycy^{-1}x^{-1} = (xy)c(xy)^{-1} = zcz^{-1},$$

and hence $a \, R \, c$ QED

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7. Variant of Saracino, Section 9, Problem 9.6:
Let $H_1 = \{e, f^2g\} = \langle f^2g \rangle$ and $H_2 = \{e, f^2\} = \langle f^2 \rangle$,
both of which are subgroups of $D_4 = \{e, f, f^2, f^3, g, fg, f^2g, f^3g\}$.

a. Find the right cosets of H_1 in D_4.
b. Find the left cosets of H_1 in D_4.
c. Find the right cosets of H_2 in D_4.
d. Find the left cosets of H_2 in D_4.
e. Briefly: what do you notice about the answers in parts (a), (b),
as compared with the answers in parts (c), (d)?

Solution. **(a)**: (Right cosets of H_1): For starters, $H_1e = H_1(f^2g) = \{e, f^2g\}$.
Multiplying by f, and computing $(f^2g)f = f^2(f^{-1}g) = fg$, we have $H_1f = H_1(fg) = \{f, fg\}$.
Multiplying by f^2, and computing $(f^2g)f^2 = f^2(f^{-2}g) = g$, we have $H_1f^2 = H_1g = \{f^2, g\}$.
Multiplying by f^3, and computing $(f^2g)f^3 = f^2(f^{-3}g) = f^3g$, we have $H_1f^3 = H_1(f^3g) = \{f^3, f^3g\}$.

Thus, $\boxed{\text{the right cosets of } H_1 \text{ are } \{e, f^2g\},\ \{f, fg\},\ \{f^2, g\},\ \{f^3, f^3g\}}$

(b): (Left cosets of H_1): For starters, $eH_1 = (f^2g)H_1 = \{e, f^2g\}$.
Multiplying by f, and computing $f(f^2g) = f^3g$, we have $fH_1 = f^3gH_1 = \{f, f^3g\}$.
Multiplying by f^2, and computing $f^2(f^2g) = g$, we have $f^2H_1 = gH_1 = \{f^2, g\}$.
Multiplying by f^3, and computing $f^3(f^2g) = fg$, we have $f^3H_1 = (fg)H_1 = \{f^3, fg\}$.

Thus, $\boxed{\text{the left cosets of } H_1 \text{ are } \{e, f^2g\},\ \{f, f^3g\},\ \{f^2, g\},\ \{f^3, fg\}}$

(c): (Right cosets of H_2): For starters, $He = H(f^2) = \{e, f^2\}$.
Multiplying by f, and computing $(f^2)f = f^3$, we have $Hf = Hf^3 = \{f, f^3\}$.
Multiplying by g, we have $Hg = H(f^2g) = \{g, f^2g\}$.
Multiplying by fg, and computing $(f^2)(fg) = f^3g$, we have $H(fg) = H(f^3g) = \{fg, f^3g\}$.

Thus, $\boxed{\text{the right cosets of } H_2 \text{ are } \{e, f^2\},\ \{f, f^3\},\ \{g, f^2g\},\ \{fg, f^3g\}}$

(d): (Left cosets of H_2): For starters, $eH = (f^2)H = \{e, f^2\}$.
Multiplying by f, and computing $f(f^2) = f^3$, we have $fH = f^3H = \{f, f^3\}$.
Multiplying by g,and computing $gf^2 = f^{-2}g = f^2g$, we have $gH = (f^2g)H = \{g, f^2g\}$.
Multiplying by fg, and computing $fg(f^2) = f(f^{-2})g = f^3g$, we have $(fg)H = (f^3g)H = \{fg, f^3g\}$.

Thus, $\boxed{\text{the left cosets of } H_2 \text{ are } \{e, f^2\},\ \{f, f^3\},\ \{g, f^2g\},\ \{fg, f^3g\}}$

(e): $\boxed{\text{For } H_1\text{, the right cosets and left cosets were different sets, but for } H_2 \text{ they are the same}}$