Homework #20

Due **Friday, May 1** in Gradescope by **11:59 pm ET**

READ Section 19 in Saracino

WATCH 1. Optional: Video 40: Reduction of Polynomials (8:29)

2. Optional: Video 41: Proof of Eisenstein (24:37)

3. Required: Video 42: Maximal and Irreducible (12:53)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (20 points) Saracino, Section 19, Problem 19.2, variant
For each of the following polynomials, determine whether or not they are irreducible in $\mathbb{Q}[X]$. If they are reducible, factor them into a product of irreducibles. [As always, justify your answers.]

(a) $X^3 - X^2 + 36$

(b) $2X^3 - 8X^2 - 6X + 20$

(c) $2X^4 + 3X^3 + 15X + 6$

Problem 2. (15 points) Saracino, Section 19, Problem 19.3, variant
For each of the following polynomials, determine whether or not they are irreducible in $\mathbb{F}_3[X]$. If they are reducible, factor them into a product of irreducibles. [As always, justify your answers.]

(a) $X^4 + X^3 + 2X^2 + X + 2$

(b) $X^4 + 2X^2 + X + 2$

Problem 3. (15 points) Saracino, Section 19, Problem 19.12
Let R be a commutative ring, and let $r \in R$. Define $\varphi_r : R[X] \to R$ by $\varphi_r(f) = f(r)$.
Prove that φ_r is a ring homomorphism.

[**Note**: This function φ_r is called an **evaluation homomorphism**. There really is something to prove here. For example, according to the definitions of $+$ and $\cdot$ in $R[X]$ on page 192, if we write $f(X) = \sum a_i X^i$ and $g(X) = \sum b_i X^i$, and we define $h(X) = f(X) + g(X)$, then

$$h(r) = (a_0 + b_0) + (a_1 + b_1)r + (a_2 + b_2)r^2 + \cdots , \quad \text{whereas}$$
$$f(r) + g(r) = (a_0 + a_1 r + a_2 r^2 + \cdots) + (b_0 + b_1 r + b_2 r^2 + \cdots),$$

which are not the exact same formulas. Your job is to use the ring properties of R to prove that they are equal, as elements of R. Then do the same thing for multiplication, which is a bit more complicated, and where you will also need to use the fact that R is a **commutative** ring.]

Problem 4. (5 points) [Not from Saracino, but may be useful on the next problem.]
Let R be a ring, let $k \geq 0$ be an integer, and let $a_0, \ldots, a_{k+1} \in R$ and $b_0, \ldots, b_{k+1} \in R$. Prove that

$$\sum_{i=0}^{k} (i+1) a_{i+1} b_{k-i} + \sum_{i=0}^{k} (k+1-i) a_i b_{k+1-i} = (k+1) \sum_{i=0}^{k+1} a_i b_{k+1-i}$$

(Problems continue on next page)

Problem 5. (20 points) Saracino, Section 19, Problem 19.17, variant
Let R be a ring. For $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, define the *formal derivative* $f'(X)$ by

$$f'(X) = a_1 + 2a_2X + 3a_3X^2 + \cdots + na_nX^{n-1},$$

or in Sigma notation,

$$f'(X) = \sum_{i\geq 0}(i+1)a_{i+1}X^i.$$

(a) For $f, g \in R[X]$, define $h = f + g$. Prove that $h'(X) = f'(X) + g'(X)$.

(b) For $f, g \in R[X]$, define $k = fg$. Prove that $k'(X) = f(X)g'(X) + f'(X)g(X)$.

(c) Assume that R is commutative. Let $n \geq 1$ be a positive integer.
Prove that the formal derivative of $[f(X)]^n$ is $n[f(X)]^{n-1} \cdot f'(X)$.

[**Suggestion**: For part (b), Problem 4 may come in handy.]

[**Note**: For part (c), please do induction on $n \geq 1$, using part (b).]

Optional Challenges (do NOT hand in): Saracino Problems 19.2(f), 19.7, 19.15, 19.18

Questions? You can ask in:

Class: MWF 11:35am – 12:25pm, SMUD 207

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

David Metacarpa's QCenter Hours, in SMUD 208:
Drop-in Hours: Mon-to-Fri, 9am – noon.
Also available by appointment in the afternoons

Math Fellow Drop-in Hours, in SMUD 206:

Sun	7:30–9:00pm	(Javier)
Mon	6:00–7:30pm	(Megan)
Tue	6:00–7:30pm	(Torin)
Tue	7:30–9:00pm	(Javier)
Wed	7:30–9:00pm	(Megan)
Thu	6:00–7:30pm	(Torin)

Also, you may email me any time at `rlbenedetto@amherst.edu`