

Homework #19Due **Tuesday, April 28** in Gradescope by **11:59 pm ET****READ** Sections 18–19 in Saracino**WATCH** 1. Required: Video 38: Polynomial Terminology (20:42)

2. Required: Video 39: Proving Irreducibility (16:38)

WRITE AND SUBMIT solutions to the following problems.**Problem 1.** (24 points) Saracino, Section 18, Problem 18.6Prove that the ring $R = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ has precisely two automorphisms.That is, find two different isomorphism $f_1, f_2 : R \rightarrow R$ (and prove they are both isomorphisms), and then prove that any arbitrary isomorphism $\varphi : R \rightarrow R$ must equal either f_1 or f_2 .[Note from RLB: you may take my word for it that R is a ring. Also, by Theorem 18.2(i), any automorphism $\varphi : R \rightarrow R$ must have $\varphi(1) = 1$. The real question is: what are the options for what $\varphi(\sqrt{2})$ could be?]**Problem 2.** (6 points) Saracino, Section 18, Problem 18.15Let $\varphi : R \rightarrow S$ be a (ring) homomorphism. Prove that φ is one-to-one if and only if $\ker \varphi = \{0_R\}$.

[Note from RLB: You may want to simply prove this if-and-only-if statement directly; the proof is not very long. But you may also be able to find an even quicker proof by quoting the corresponding fact about group homomorphisms. If you do it that second way, be sure you are clear about why all the claims you make along the way are legitimate.]

Problem 3. (14 points) Saracino, Section 18, Problem 18.22(a)Let $\varphi : R \rightarrow S$ be a (ring) homomorphism, and let J be a prime ideal of S . If $\varphi^{-1}(J) \neq R$, prove that $\varphi^{-1}(J)$ is a prime ideal of R .[Note from RLB: Yes, Theorem 18.4(iii) gives the “ideal” part of this, but since that theorem has no proof provided, please prove *both* that $\varphi^{-1}(J)$ is an ideal of R , *and* that it is prime.]**Problem 4.** (14 points) Saracino, Section 18, Problem 18.28In the proof of Theorem 18.10, we had an integral domain D and a set F called the field of fractions of D . Prove that the operations $+$ and \cdot defined on F in that proof are well-defined.**Problem 5.** (12 points) Saracino, Section 19, Problem 19.1Let $f(X) = a_0 + a_1X + \cdots + a_rX^r \in \mathbb{Z}[X]$. Suppose $m/n \in \mathbb{Q}$, with $(m, n) = 1$. If m/n is a root of f , prove that $m \mid a_0$ and $n \mid a_r$.**Optional Challenges (do NOT hand in):** Saracino Problems 17.34, 18.24, 18.29

Questions? You can ask in:

Class: MWF 11:35am – 12:25pm, SMUD 207

My office hours: in my office (SMUD 406):

Mon 2:00–3:30pm

Tue 1:30–3:15pm

Fri 1:00–2:00pm

David Metacarpa’s QCenter Hours, in SMUD 208:

Drop-in Hours: Mon-to-Fri, 9am – noon.

Also available by appointment in the afternoons

Math Fellow Drop-in Hours, in SMUD 206:

Sun 7:30–9:00pm (Javier)

Mon 6:00–7:30pm (Megan)

Tue 6:00–7:30pm (Torin)

Tue 7:30–9:00pm (Javier)

Wed 7:30–9:00pm (Megan)

Thu 6:00–7:30pm (Torin)

Also, you may email me any time at rlbenedetto@amherst.edu