Homework #17

Due **Friday, April 17** in Gradescope by **11:59 pm ET**

READ Sections 16–17 in Saracino and the "Basic Definitions" handouts on **Rings** and **Ideals**

WATCH 1. Required: Video 32: Types of Rings (8:20)

2. Optional: Video 33: Subrings (11:03)

3. Required: Video 34: Proofs on Prime Ideals (17:04)

WRITE AND SUBMIT solutions to the following problems.

Problem 1. (16 points) Saracino, Section 16, Problem 16.3:
Let $F = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Prove that F is a field under ordinary addition and multiplication.

[**Note from RLB**: Even though they are in a later section, you may use either Theorem 17.1 or Corollary 17.2 to help prove that $F \subseteq \mathbb{R}$ is a ring. For the remaining field axioms, the hardest is the existence of multiplicative inverses, for which you may assume the fact that $\sqrt{2} \notin \mathbb{Q}$.]

Problem 2. (6 points) Saracino, Section 16, Problem 16.7:
Let F be a field, let $a, b \in F$, and assume $a \neq 0$. Show that the equation $ax + b = 0$ can be solved for $x \in F$; that is, there exists $x \in F$ that makes the equation true.

Problem 3. (11 points) Saracino, Section 16, Problem 16.18, slight variant:
Let R be a nontrivial ring with unity (so $1 \neq 0$), and assume that R has no nonzero zero-divisors. Let $a, b \in R$ with $ab = 1$. Prove that $ba = 1$ also.

[**Warning from RLB**: you can't refer to a^{-1} or b^{-1} in your proof. Until you have actually shown that $ba = 1$, we don't know that either a or b has a multiplicative inverse at all.]

Problem 4. (16 points) Saracino, Section 16, Problem 16.24, variant:
Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, where i is the complex number $i = \sqrt{-1}$.
It is a fact, which you may assume without proof, that $\mathbb{Z}[i]$, called the *ring of Gaussian integers*, is a commutative ring with unity. For any $r = a + bi \in \mathbb{Z}[i]$, define the *norm* $N(r)$ by $N(r) = a^2 + b^2$.

(a) Prove that for all $r, s \in \mathbb{Z}[i]$, we have $N(rs) = N(r)N(s)$.

(b) Use part (a) to prove that $r \in \mathbb{Z}[i]$ is a unit if and only if $N(r) = 1$.

(c) Use part (b) to find all the units in $\mathbb{Z}[i]$. (And (briefly) justify your answer, of course.)

Problem 5. (12 points) Saracino, Section 17, Problem 17.2(a,c), ideals only:
Let $R = \{f : \mathbb{R} \to \mathbb{R}\}$ be the ring of real-valued functions on the real line, under ordinary addition and multiplication of functions. Which of the following subsets S of R are ideals?
[As always, prove your answers.]

(a) $S = \{f \in R \mid f(1) = 0\}$

(c) $S = \{f \in R \mid f(3) = f(4)\}$

(Problems continue on next page)

Problem 6. (9 points) Saracino, Section 17, Problem 17.25(a):
Let R be a ring, and let I and J be ideals of R. Prove that $I \cap J$ is an ideal of R.

Optional Challenges (do NOT hand in): Saracino Problems 16.5, 16.22, 16.25

Questions? You can ask in:

Class: MWF 11:35am – 12:25pm, SMUD 207

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

David Metacarpa's QCenter Hours, in SMUD 208:
Drop-in Hours: Mon-to-Fri, 9am – noon.
Also available by appointment in the afternoons

Math Fellow Drop-in Hours, in SMUD 206:

Sun	7:30–9:00pm	(Javier)
Mon	6:00–7:30pm	(Megan)
Tue	6:00–7:30pm	(Torin)
Tue	7:30–9:00pm	(Javier)
Wed	7:30–9:00pm	(Megan)
Thu	6:00–7:30pm	(Torin)

Also, you may email me any time at `rlbenedetto@amherst.edu`