Homework #14

Due **Tuesday, March 31** in Gradescope by **11:59 pm ET**

READ Sections 11–12 in Saracino and the (optional!) **Cauchy's Theorem** handout

WATCH Optional: Video 28: Cauchy's Theorem for Abelian Groups (13:35)

WRITE AND SUBMIT solutions to the following problems.

Don't forget to **justify all your claims**. (But a lot of justifications are short!)

Problem 1. (12 points) Saracino, Section 11, Problem 11.20:
Let G be a group, and let $H \lhd G$ be a normal subgroup such that $[G : H] = 20$ and $|H| = 7$. Suppose that $x \in G$ and $x^7 = e$. Prove that $x \in H$.

[*Hint*: I really don't know a way of doing this problem without looking at the quotient group G/H, which is a thing that makes sense because $H \lhd G$.]

Problem 2. (15 points) Saracino, Section 11, Problem 11.30(b), slight variant:
Let G be a group. A **commutator** is an element of G that can be written as $xyx^{-1}y^{-1}$ for some $x, y \in G$. Let $C \subseteq G$ be the set of all the commutators, i.e., $C = \{xyx^{-1}y^{-1} \mid x, y \in G\}$.

Prove that for any subgroup $K \subseteq G$, the following are equivalent:

(i) $C \subseteq K$.
(ii) $K \lhd G$, and G/K is abelian.

[Note from RLB: The set C is (usually) *not* a subgroup of G, as it is usually not closed under the group operation. (I.e., the product of two commutators is usually not itself a commutator.) However, one can define the **commutator subgroup** G' of G to be the subgroup of G generated by the commutators. That is, G' is the smallest subgroup containing C; or equivalently, G' is the intersection of all the subgroups of G that contain C.]

Problem 3. (24 points) Saracino, Section 12, Problem 12.1(a,b,c,d):
Which of the following mappings are homomorphisms? For those that are, which are one-to-one (monomorphisms), which are onto (epimorphisms), and which are both (**isomorphisms**)?

(a) $G = \mathbb{R}^\times$, $H = \mathbb{R}_{>0}$, and $\varphi : G \to H$ by $\varphi(x) = |x|$.
(b) $G = \mathbb{R}_{>0}$, and $\varphi : G \to G$ by $\varphi(x) = \sqrt{x}$.
(c) $G =$ group of polynomials with real coefficients, under addition,
and $\varphi : G \to \mathbb{R}$ by $\phi(p) = p(1)$.
(d) G as in (c), and $\phi : G \to G$ by $\phi(p) = p'$, the derivative of $p(x)$.

[As always, justify any claims you make along the way, usually briefly.]

Problem 4. (7 points) Saracino, Section 12, Problem 12.3, first part:
Let G be an abelian group, let $n \geq 1$ be a positive integer, and let $\varphi : G \to G$ by $\varphi(x) = x^n$. Prove that φ is a homomorphism.

(Problems continue on next page)

Problem 5. (15 points) Saracino, Section 12, Problem 12.4(a,b):
In each case, determine whether or not the two groups are isomorphic. [And prove it!]

(a) $(C_{12}, \oplus)$ and $(\mathbb{Q}_{>0}, \cdot)$
(b) $(2\mathbb{Z}, +)$ and $(3\mathbb{Z}, +)$

Problem 6. (10 points) Saracino, Section 12, Problem 12.5:
Let G, H be groups. Prove that $G \times H \cong H \times G$.

Optional Challenges (do NOT hand in): Saracino Problems 11.23, 11.24, 11.29

Questions? You can ask in:

Class: MWF 11:35am – 12:25pm, SMUD 207

My office hours: in my office (SMUD 406):
Mon 2:00–3:30pm
Tue 1:30–3:15pm
Fri 1:00–2:00pm

David Metacarpa's QCenter Hours, in SMUD 208:
Drop-in Hours: Mon-to-Fri, 9am – noon.
Also available by appointment in the afternoons

Math Fellow Drop-in Hours, in SMUD 206:

Sun	7:30–9:00pm	(Javier)
Mon	6:00–7:30pm	(Megan)
Tue	6:00–7:30pm	(Torin)
Tue	7:30–9:00pm	(Javier)
Wed	7:30–9:00pm	(Megan)
Thu	6:00–7:30pm	(Torin)

Also, you may email me any time at `rlbenedetto@amherst.edu`