Math 350, Spring 2025 Professor Rob Benedetto

Final Exam, Take-Home

Due **Wednesday, May 14** in Gradescope by **11:59 pm ET**

(You are welcome to submit it before that, if you want.)

Instructions: Do all **eight numbered problems** (totalling 200 points), and in addition, as a ninth "problem," **write out and sign the academic honesty pledge found later in this document**. There are also two optional bonus problems worth a total of 4 points.

Write your answers neatly and legibly, and tag them correctly on gradescope.

You must fully justify your answers. Simple algebraic deductions need not be explained, but other leaps of logic require a word or two of justification. You may use theorems (and lemmas, etc.) from class or the book to do so; however, make sure you verify all the relevant hypotheses of any theorem you use, and reference the source. (E.g., "By Theorem 12.1," or "By Lagrange's Theorem," or "By HW 11, Problem 4," etc.) Unless otherwise noted, **you may quote only theorems that we covered or exercises that were assigned (NOT challenge problems, results from other sections, or other unassigned problems)**. If you are not sure whether some argument or statement requires further justification, please ask me about it.

In working on the problems, you **may** use:

- **the textbook** (Saracino, Sections 0–13 and 16–20),
- **handouts and videos from the course and the course websites**,
- **your own Math 350 notes**, and
- **your own graded Math 350 homework (on gradescope) and exams**.

But of course, as a matter of Academic Honesty, until after the exam deadline has passed:

> **You may NOT use other books, online information, AI, calculators, or any other outside sources.**
> You may NOT discuss the problems with anyone other than me.
> You may NOT share this exam document with anyone other than me.

However,

- You should feel free to talk to me about anything on this exam. I will be much less helpful than I am for homework assignments, but I will be happy to clarify things. You **may** ask me anything you want; I will decide how much I can answer.
- You **may** talk to Allison about concepts from the course, old homework problems, and other old course materials, **but you must be careful to stay away from discussing the actual exam problems with her**.

The exam is due at **11:59 PM ET** on the Wednesday of exam period, on gradescope. You may submit it early, even days early, but **no extensions will be granted**. Outside of truly exceptional circumstances, any exam not submitted on time will be graded as a zero.

I strongly recommend that you

> **consider the deadline to be 8pm ET, Wednesday, May 14**

so that you have a four-hour grace period in case any unexpected snags arise.

1. **(25 points)** Find all subgroups of $C_3 \times C_3$. That is, in your writeup:

- List all the subgroups,
- Prove that each item in your list is indeed a subgroup, and
- Prove that every subgroup does indeed appear on your list.

[*Hint*: There are 6 different subgroups in total. You may **not** assume that fact in your proofs; I just mention it to you for your convenience.]

2. **(25 points)** Consider the 200-element dihedral group

$$D_{100} = \{e, f, f^2, \ldots, f^{99}, g, fg, f^2g, \ldots, f^{99}g\}$$

of rotations and flips of a regular 100-sided polygon.
Let $h \in D_{100}$ be the flip $h = f^{18}g$, and let $H = Z(h)$ be the centralizer of h in D_{100}.
(That is, $H = \{x \in D_{100} \mid hx = xh\}$, which we know to be a subgroup, by HW 6, Problem 6.)

2a. Prove that $H = \{e, f^{50}, f^{18}g, f^{68}g\}$.

2b. Prove that H is **not** a normal subgroup of D_{100}.

3. **(21 points)** Let $\varphi : A_4 \to C_8$ be a homomorphism, where A_4 is the (12-element) alternating group on 4 symbols, and C_8 is the cyclic group of order 8.
Prove that φ is the trivial homomorphism. That is, prove that $\varphi(g) = 0$ for all $g \in A_4$.

[**Hint**: if $\sigma \in A_4$ is a 3-cycle, what can you prove about $\varphi(\sigma)$?]

4. **(15 points)** Let G_1, G_2, G_3 be groups, and let $\varphi : G_1 \to G_2$ and $\psi : G_2 \to G_3$ be homomorphisms. Suppose that ψ is onto, and that $\psi \circ \varphi$ is the trivial homomorphism.

4a. Prove that $\ker \psi \supseteq \varphi(G_1)$.

4b. If $\ker \psi = \varphi(G_1)$, prove that $G_3 \cong G_2/\varphi(G_1)$.

5. **(23 points)** Let R be a ring with unity. Define a relation $\sim$ on R by, for $a, b \in R$:

$$a \sim b \iff \exists u \in R^\times \text{ s.t. } b = au.$$

5a. Prove that $\sim$ is an equivalence relation on R.

5b. Suppose further that R is an integral domain.
For any $a, b \in R$, prove that $aR = bR$ if and only if $a \sim b$.
[Recall that aR denotes the principal ideal generated by a.]

6. **(21 points)** Let $\varphi : R \to S$ be an **onto** homomorphism of rings, and let $I \subseteq R$ be a **prime** ideal. By Theorem 18.4(iv), we know that $\varphi(I)$ is an ideal of S.
Suppose that $\ker \varphi \subseteq I$. Prove that $\varphi(I)$ is a prime ideal of S.

(Problems continue on next page)

7. **(35 points)** Let R be a commutative ring, and let $I \subseteq R$ be an ideal. Define

$$J = \{r \in R \mid \text{there is an integer } n \geq 1 \text{ such that } r^n \in I\}.$$

7a. Let $x, y \in R$, and let $k \geq 1$ be a positive integer. Prove that there are integers $c_0, \ldots, c_k \in \mathbb{Z}$ such that $(x-y)^k = c_0 x^k + c_1 x^{k-1} y + c_2 x^{k-2} y^2 + \cdots + c_{k-1} x y^{k-1} + c_k y^k$. [*Suggestion*: Use induction on k. You do *not* need to find a formula for the integers c_i.]

7b. Prove that J is an ideal of R.

[*Suggestion*: Part (a) may come in handy at some point.]

7c. Prove that the quotient ring R/J contains no nonzero nilpotent elements.

8. **(35 points)** Let $\mathbb{F}_2 = \{0, 1\}$ denote the field of two elements (that the book calls $\mathbb{Z}_2$). Let $f = X^4 + X + 1 \in \mathbb{F}_2[X]$.

8a. Prove that f is irreducible in $\mathbb{F}_2[X]$.

8b. Use f and the ideas of Section 20 to construct a field with exactly 16 elements.

Don't forget to justify all of your claims. (As usual, you may quote theorems to do so, but for example, in part (b) you must prove that the object you construct is indeed a field, and that it has exactly 16 elements.)

9. (**Absolutely required**): Write out the following statement and sign your name to it:

I have read, understood, and followed the Academic Honesty instructions on the cover page of this Final Exam.

OPTIONAL BONUS A. **(2 points)** Recall that A_6 denotes the alternating group on 6 objects, and S_4 is the symmetric group on 4 objects. Find an **injective** homomorphism $\varphi : S_4 \to A_6$. (And of course, prove all your claims.)

OPTIONAL BONUS B. **(2 points)** Let $R = \mathbb{Z}[\sqrt{11}] = \{a + b\sqrt{11} \mid a, b \in \mathbb{Z}\}$, which is a subring of $\mathbb{R}$. Find a unit $u \in R^\times$ of infinite order in the group of units $R^\times$. (And of course, prove all your claims.)