Math 350-01, Spring 2026 Professor Rob Benedetto

Final Exam, Take-Home
Due **Wednesday, May 13** in Gradescope by **11:59 pm ET**
(You are welcome to submit it before that, if you want.)

Instructions: Do all **eight numbered problems** (totalling 200 points), and in addition, as a ninth "problem," **write out and sign the academic honesty pledge found later in this document**. There are also two optional bonus problems worth a total of 4 points.

Answers must be written neatly and legibly, and matched to the correct problem numbers on Gradescope.

You must fully justify your answers. Simple algebraic deductions need not be explained, but other steps require a word or two of justification. You may use theorems (and lemmas, etc.) from class or the book to do so; however, make sure you verify all the relevant hypotheses of any theorem you use, and reference the source. (E.g., "By Theorem 12.1," or "By Lagrange's Theorem," or "By HW 11, Problem 4," etc.) Unless otherwise noted, **you may quote only theorems that we covered or exercises that were assigned (NOT results from other sections, challenge problems, or other unassigned problems)**. If you are not sure whether some argument or statement requires further justification, please ask me about it.

In working on the problems, you **may** use:

- **the textbook** (Saracino, Sections 0–13 and 16–20),
- **handouts and videos from the course and the course websites**,
- **your own Math 350 notes**, and
- **your own Math 350 homework and exams**.

But of course, as a matter of Academic Honesty, until after the exam deadline has passed:

You may NOT use other books, online information, AI tools, calculators, or any other outside sources.
You may NOT discuss the problems with anyone other than me.
You may NOT share this exam document with anyone other than me.

However,

- You should feel free to talk to me about anything on this exam. I will probably be much less helpful than I am for homework assignments, but I will be happy to clarify things. You **may** ask me anything you want; I will decide how much I can answer.
- You **may** talk to Allison Tanguay or David Metacarpa about concepts from the course, old homework problems, and other old course materials, **but you must be careful to stay away from discussing the actual exam problems with them**.

The exam is due at **11:59 PM ET** on the Wednesday of exam period, on Gradescope. You may submit it early, even days early, but **no extensions will be granted**. Outside of truly exceptional circumstances, any exam not submitted on time will be graded as a zero.

I strongly recommend that you

plan to submit the exam before 8pm ET, Wednesday, May 13

so that you have a four-hour grace period in case any unexpected snags arise.

1. **(15 points)** Let G_1 and G_2 be groups, and let $\varphi : G_1 \to G_2$ be a homomorphism with kernel $N = \ker \varphi$. Let $a, b \in G_1$. Prove that

$$\varphi(a) = \varphi(b) \iff Na = Nb$$

2. **(20 points)** For each of the following groups G, decide whether G has a subgroup of order 6. If it does, give an example of such a subgroup, and prove that your example is indeed a subgroup of G. If G doesn't have such a subgroup, prove that it doesn't.

2a. C_{99} (which is what the book calls $\mathbb{Z}_{99}$)
2b. $C_3 \times C_{15}$
2c. $C_9 \times Q_8$
2d. S_5

3. **(25 points)** Consider the 18-element group D_9, of rotations and flips of a regular 9-gon.

3a. Prove that D_9 does *not* have any elements of order 6.
3b. Find a subgroup $H \subseteq D_9$ of order 6. [And prove that H actually *is* a subgroup.]

4. **(20 points)** Let $n \geq 2$ be an integer, let G be a group of odd order, and let $\varphi : S_n \to G$ be a homomorphism.

4a. Let $\tau \in S_n$ be a transposition. Prove that $\varphi(\tau) = e$.
4b. For *any* $\sigma \in S_n$, prove that $\varphi(\sigma) = e$
(*Suggestion:* use part (a).)

5. **(35 points)** Let G be an abelian group, and define

$$H = \{x \in G \mid x^{2^n} = e \text{ for some integer } n \geq 0\}.$$

Equivalently, an element $x \in G$ belongs to H if and only if $o(x)$ is a power of 2.

5a. Prove that H is a subgroup of G.
[**Note**: Once you have proven that, then because G is abelian, it then follows that H is normal in G.]
5b. Let $y \in G$ and let $m \geq 1$ be an integer. Suppose that $(Hy)^{2m}$ is the identity of G/H. Prove that $(Hy)^m$ is already the identity.
5c. Use part (b) to prove that no element of G/H has even order.

(Problems continue on next page)

6. **(20 points)** Let R be a ring, and let $I \subseteq R$ be an ideal. Prove that

$$R/I \text{ is commutative} \qquad \Longleftrightarrow \qquad \forall x, y \in R, \quad xy - yx \in I.$$

7. **(30 points)** Let R be a commutative ring with unity, and let $I \subseteq R$ be an ideal. Prove that **the following two statements are equivalent**:

(i) $R^{\times} = R \setminus I$. (That is, the set of units of R is precisely the complement of I.)

(ii) I is a maximal ideal of R, and every proper ideal of R is contained in I.

8. **(35 points)** Let $\mathbb{F}$ be a finite field with $q \geq 2$ elements, and let $f = X^2 + aX + b \in \mathbb{F}[x]$ be a monic polynomial of degree 2. Let $I = \langle f \rangle$ be the principal ideal of $\mathbb{F}[X]$ generated by f; that is,

$$I = \langle f \rangle = \{fg \mid g \in \mathbb{F}[X]\}.$$

Let $R = \mathbb{F}[X]/I$ be the resulting quotient ring.

8a. Prove that $|R| = q^2$. That is, prove that R has exactly q^2 elements.

8b. If $f = X^2$, prove that R has at least one nonzero nilpotent element.

8c. Still assuming $f = X^2$, prove that every zero-divisor in R is nilpotent.

9. (**Absolutely required**): Write out the following statement and sign your name to it:

I have read, understood, and followed the Academic Honesty instructions on the cover page of this Final Exam.

BONUS A. **(2 points)** Prove that S_7 contains a nonabelian subgroup of order 21.

BONUS B. **(2 points)** Let $R = M_2(\mathbb{F}_2)$, the ring of 2×2 matrices with entries in the field $\mathbb{F}_2$ (which is the field the book calls $\mathbb{Z}_2$), under usual matrix addition and matrix multiplication.

The group of units $R^{\times}$ is isomorphic to one of the standard groups we know. (Like C_n, D_n, S_n, A_n, etc.). Decide which one it is, and prove that $R^{\times}$ is indeed isomorphic to that group.