

Solutions to Midterm Exam 2

1. (**23 points**) Consider the group S_8 of permutations of the set $\{1, 2, 3, \dots, 8\}$. Let $\sigma, \tau \in S_8$ be the permutations

$$\sigma = (1, 2, 3)(6, 7, 8) \quad \text{and} \quad \tau = (1, 3, 5, 6)(2, 8)(4, 7)$$

Don't forget to (briefly) justify your answers in (b), (c), (d) below:

- Write $\sigma\tau$ as a product of **disjoint** cycles. (There's not much justification to show here.)
- Compute the **order** of each of σ , τ , and $\sigma\tau$.
- Decide whether each of σ , τ , and $\sigma\tau$ is an **even** or **odd** permutation.
- Find an element of S_8 of order 15.

Solution. (a): $\sigma\tau = (1, 2, 3)(6, 7, 8)(1, 3, 5, 6)(2, 8)(4, 7) = (1)(2, 6)(3, 5, 7, 4, 8) = \boxed{(2, 6)(3, 5, 7, 4, 8)}$

(b): $o(\sigma) = \text{lcm}(3, 3) = \boxed{3}$ $o(\tau) = \text{lcm}(4, 2, 2) = \boxed{4}$ $o(\sigma\tau) = \text{lcm}(2, 5) = \boxed{10}$

(c): 3-cycles are even, so σ is even + even = **even**

Similarly, τ is odd + odd + odd = **odd**

And $\sigma\tau$ is odd + even = **odd**

(d): The order of $\boxed{(1, 2, 3)(4, 5, 6, 7, 8) \in S_8}$ is $\text{lcm}(3, 5) = 15$

2. (**20 points**) Let G be a group, and let $H, K \subseteq G$ be subgroups. Define a relation R on G as follows: for any $a, b \in G$:

$$a R b \quad \text{means} \quad \exists h \in H \text{ and } \exists k \in K \text{ such that } a = hbk.$$

Prove that R is an equivalence relation on G .

Proof. (Reflexive): Given $a \in G$, we have $a = eae$. Since $e \in H$ and $e \in K$, we have aRa .

(Symmetric): Given $a, b \in G$ such that aRb , there exist $h \in H$ and $k \in K$ such that $a = hbk$. Then $h^{-1}ak^{-1} = b$. Since $h^{-1} \in H$ and $k^{-1} \in K$, we have bRa .

(Transitive): Given $a, b, c \in G$ such that aRb and bRc , there exist $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $a = h_1bk_1$ and $b = h_2ck_2$. Then $a = h_1(h_2ck_2)k_1 = (h_1h_2)c(k_2k_1)$.

Since $h_1h_2 \in H$ and $k_2k_1 \in K$, it follows that aRc .

QED

3. (**17 points**) Define $\varphi : \mathbb{R} \rightarrow GL(2, \mathbb{R})$ by $\varphi(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$.

Prove that φ is an injective homomorphism.

Proof. Homom: Given $x, y \in \mathbb{R}$, we have

$$\varphi(x)\varphi(y) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ 0 & 1 \end{bmatrix} = \varphi(x+y),$$

as desired.

Injective: Given $x, y \in \mathbb{R}$ such that $\varphi(x) = \varphi(y)$, then $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}$, and hence $x = y$. QED

Alternate Proof of Injective: Given $x \in \ker \varphi$, we have $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} = \varphi(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and hence $x = 0$.

Thus, $\ker \varphi = \{0\}$. and therefore φ is injective.

QED

4. (20 points) Let G be a group, and let $H \triangleleft G$ and $K \triangleleft G$ be normal subgroups of G . Let $a \in H$ and $b \in K$.

4a. Prove that $aba^{-1}b^{-1} \in H \cap K$.

4b. Suppose further that $|H| = 8$ and $|K| = 15$. Prove that $ab = ba$.

Proof. (a): Since $b \in K$ and $a \in H \subseteq G$, and since $K \triangleleft G$, we have $aba^{-1} \in K$. Therefore, since $b^{-1} \in K$ as well, we have $aba^{-1}b^{-1} = (aba^{-1})b^{-1} \in K$.

Similarly, since $a^{-1} \in H$ and $b \in K \subseteq G$, and since $H \triangleleft G$, we have $ba^{-1}b^{-1} \in K$. Therefore, since $a \in H$ as well, we have $aba^{-1}b^{-1} = a(ba^{-1}b^{-1}) \in H$.

Thus, we have shown $aba^{-1}b^{-1} \in H \cap K$.

QED (a)

(b): Recall that $H \cap K$ is a subgroup of G , and hence, being a group that is a subset of both H and K , it is a subgroup of both H and K .

Let $m = |H \cap K|$. By Lagrange's Theorem, we have $m \mid |H|$ and $m \mid |K|$. That is, $m \mid 8$ and $m \mid 15$. Thus, $m \mid \gcd(8, 15) = 1$, and hence $|H \cap K| = m = 1$. Therefore, $H \cap K = \{e\}$.

By part (a), we have $aba^{-1}b^{-1} \in H \cap K$, and hence $aba^{-1}b^{-1} = e$. Thus, $ab = ba$.

QED (b)

5. (20 points) Let $G = C_4 \times C_2$, and let $H \subseteq G$ be the cyclic subgroup $H = \langle (2, 0) \rangle$.

5a. List all of the elements of H .

5b. List all of the elements of the quotient group G/H .

Make sure your answer makes clear how many **distinct** elements there are.

5c. Compute the order of the element $H + (1, 1)$ of G/H .

Solution/Proof. (a): We have $(2, 0) + (2, 0) = (4, 0) = (0, 0)$, so $H = \{(0, 0), (2, 0)\}$

(b): We have $H + (1, 0) = \{(1, 0), (3, 0)\}$, and $H + (1, 1) = \{(1, 1), (3, 1)\}$, and $H + (0, 1) = \{(0, 1), (2, 1)\}$. Together with $H + (0, 0) = H$, these cosets account for all 8 elements of G .

Thus, G/H is the four-element group $\{H + (0, 0), H + (1, 0), H + (1, 1), H + (0, 1)\}$

(c): We have $H + (1, 1) \neq H + (0, 0)$ since $(1, 1) - (0, 0) = (1, 1) \notin H$.

However, $2(H + (1, 1)) = H + (1, 1) + (1, 1) = H + (2, 0) = H + (0, 0)$,

since $(2, 0) - (0, 0) = (2, 0) \in H$.

Thus, $o(H + (1, 1)) = 2$

OPTIONAL BONUS. (2 points.) Recall that D_{25} denotes the dihedral group of order 50 (rotations and flips of a regular 25-gon), and C_{25} denotes the cyclic group of order 25. Let $\varphi : D_{25} \rightarrow C_{25}$ be a homomorphism. Prove that $\varphi(x) = 0$ for all $x \in D_{25}$.

Proof. Let $a = \varphi(g) \in C_{25}$ and $b = \varphi(fg) \in C_{25}$.

Then $2a = \varphi(g^2) = \varphi(e) = 0$ and $2b = \varphi((fg)^2) = \varphi(e) = 0$.

Therefore, $a = 26a = 13(2a) = 13(0) = 0$, and $b = 26b = 13(2b) = 13(0) = 0$, where the first equality in each case is because $a, b \in C_{25}$.

Thus, we have shown that $\varphi(g) = 0$ and $\varphi(fg) = 0$.

It follows that $\varphi(f) = \varphi((fg)g) = \varphi(fg) + \varphi(g) = 0 + 0 = 0$.

Hence, for any $x \in D_{25}$, we have $x = f^i g^j$ for some $i, j \in \mathbb{Z}$, and therefore

$\varphi(x) = i\varphi(f) + j\varphi(g) = i(0) + j(0) = 0$.

QED