Solutions to Practice Problems for Midterm Exam 2

1.(a) $\displaystyle\int_{|z|=6} \frac{\cos(\pi z)}{(z-1)^3}\,dz$

Answer. Let $D = D(0,6)$, and $f(z) = \cos(\pi z)$, which is analytic everywhere, and in particular on $\overline{D}$. We have $f'(z) = -\pi \sin \pi z$ and hence $f''(z) = -\pi^2 \cos \pi z$. Since $1 \in D$, by CDF [Cauchy Differentiation Formula] the integral is $\displaystyle\int_{\partial D} \frac{f(z)}{(z-1)^3}\,dz = \frac{2\pi i}{2!} f''(1) = \pi i \cdot (-\pi^2 \cos \pi) = \pi^3 i$.

1.(b) $\displaystyle\int_{|z|=2} \frac{e^{i\pi z/2}}{z(z+1)(z+3)}\,dz$

Answer. First, let $D = D(0,2) \smallsetminus [\overline{D}(0,1/3) \cup \overline{D}(-1,1/3)]$, and let $f(z)$ be the integrand of this integral. Then f is analytic on $\overline{D}$, and hence $\int_{\partial D} f(z)\,dz = 0$. Since ∂D consists of three circles, with the inner ones traced in the negative direction, we have

$$\int_{|z|=2} f(z)\,dz = \int_{|z|=1/3} f(z)\,dz + \int_{|z+1|=1/3} f(z)\,dz.$$

For the first of these two integrals, use $g(z) = e^{i\pi z/2}/[(z+1)(z+3)]$, which is analytic on $\overline{D}(0,1/3)$. By CIF [Cauchy Integral Formula], this first integral is $2\pi i g(0) = 2\pi i/3$.
For the second, use $h(z) = e^{i\pi z/2}/[z(z+3)]$, which is analytic on $\overline{D}(-1,1/3)$. By CIF [Cauchy Integral Formula], this second integral is $2\pi i h(-1) = 2\pi i e^{-i\pi/2}(-1/2) = -\pi$.
So the original integral is $-\pi + 2\pi i/3$.

1.(c) $\displaystyle\int_{|z-3|=2} \frac{\operatorname{Log} z}{z^2(z-4)^2}\,dz$

Answer. Let $D = D(3,2)$, and $f(z) = \operatorname{Log} z/z^2$, which is analytic on $\mathbb{C} \smallsetminus (-\infty, 0]$ and hence on $\overline{D}$. We have $f'(z) = [(1/z)z^2 - 2z \operatorname{Log} z]/z^4 = (1 - 2\operatorname{Log} z)/z^3$. Since $4 \in D$, by CDF the integral is $\displaystyle\int_{\partial D} \frac{f(z)}{(z-4)^2}\,dz = \frac{2\pi i}{1!} f'(4) = 2\pi i\Big(\frac{1-2\log 4}{4^3}\Big) = \frac{\pi i(1-4\log 2)}{32}$. [And yes, $2\log 4 = 4\log 2$, although you can leave it as $2\log 4$ if you want.]

1.(d) $\displaystyle\int_{|z|=4} \frac{e^{5z}}{(z-\pi i)^3}\,dz$

Answer. Let $D = D(0,4)$, and $f(z) = e^{5z}$, which is analytic on $\mathbb{C}$ and hence on $\overline{D}$. We have $f'(z) = 5e^{5z}$ and $f''(z) = 25e^{5z}$. Since $\pi i \in D$, by CDF the integral is
$\displaystyle\int_{\partial D} \frac{f(z)}{(z-\pi i)^3}\,dz = \frac{2\pi i}{2!} f''(\pi i) = \pi i(25e^{5\pi i}) = -25\pi i$.

1.(e) $\displaystyle\int_{|z|=\pi} \frac{e^{5z}}{(z-4)^4}\,dz$

Answer. Let $D = D(0,\pi)$, and $f(z) = e^{5z}/(z-4)^4$, which is analytic on $\mathbb{C} \smallsetminus \{4\}$ and hence on $\overline{D}$. [Note: $4 \notin \overline{D}$!!!] So by Cauchy's Theorem, the integral is 0.

1.(f) $\displaystyle\int_{|z-5|=4} \frac{(z-3)\sin z}{z^3(z-6)(z-8)}\,dz$

Answer. Let $D = D(5,4) \setminus [\overline{D}(6,1/2) \cup \overline{D}(8,1/2)]$, and let $f(z)$ be the integrand of this integral. Then f is analytic on $\overline{D}$, and hence $\int_{\partial D} f(z)\,dz = 0$. Since ∂D consists of three circles, with the inner ones traced in the negative direction, we have

$$\int_{|z-5|=4} f(z)\,dz = \int_{|z-6|=1/2} f(z)\,dz + \int_{|z-8|=1/2} f(z)\,dz.$$

For the first of these two integrals, use $g(z) = (z-3)\sin z/[z^3(z-8)]$, which is analytic on $\overline{D}(6,1/2)$. By CIF [Cauchy Integral Formula], this first integral is $2\pi i g(6) = 2\pi i(3\sin 6)/(6^3 \cdot (-2)) = -i\pi \sin 6/72$ For the second, use $h(z) = (z-3)\sin z/[z^3(z-6)]$, which is analytic on $\overline{D}(8,1/2)$. By CIF [Cauchy Integral Formula], this second integral is $2\pi i h(8) = 2\pi i(5\sin 8)/(8^3 \cdot 2) = 5i\pi \sin 8/512$.
So the original integral is $i\pi\Big(\dfrac{5\sin 8}{512} - \dfrac{\sin 6}{72}\Big)$.

---

2. For each of the following functions, find its full power series expansion about $z = 0$, as well as the radius of convergence of this power series.

Answers. (a) $f(z) = z\,\mathrm{Log}(z+2)$:
First, we note that $\dfrac{1}{z+2} = \dfrac{1}{2} \cdot \dfrac{1}{1-(-z/2)} = \sum_{k\geq 0} \dfrac{(-1)^k}{2^{k+1}} z^k$, with radius of convergence $R = 2$, since $1/(z+2)$ is analytic on $\mathbb{C} \setminus \{-2\}$ but blows up at -2, and $|-2-0| = 2$.
Antidifferentiating, $\mathrm{Log}(z+2) = C + \sum_{k\geq 0} \dfrac{(-1)^k}{2^{k+1}(k+1)} z^{k+1} = C + \sum_{k\geq 1} \dfrac{(-1)^{k-1}}{k \cdot 2^k} z^k$ for some constant C, with the same radius of convergence $R = 2$.
Plugging $z = 0$ into both sides, we have $C = \mathrm{Log}\,2$. Finally, multiplying by z (which does not change R), we have $f(z) = z\log 2 + \sum_{k\geq 2} \dfrac{(-1)^k}{(k-1)\cdot 2^{k-1}} z^k$, with radius of convergence $R = 2$.

(b) $g(z) = \dfrac{z^2}{(z^5-4)^3}$:
First, we note that $\dfrac{1}{z-4} = \dfrac{-1}{4} \cdot \dfrac{1}{1-(z/4)} = \sum_{k\geq 0} \dfrac{-1}{4^{k+1}} z^k$, with radius of convergence $R = 4$, since $1/(z-4)$ is analytic on $\mathbb{C} \setminus \{4\}$ but blows up at 4, and $|4-0| = 4$.
Differentiating, $\dfrac{-1}{(z-4)^2} = \sum_{k\geq 0} \dfrac{-k}{4^{k+1}} z^{k-1}$.
Differentiating again, $\dfrac{2}{(z-4)^3} = \sum_{k\geq 0} \dfrac{-k(k-1)}{4^{k+1}} z^{k-2} = \sum_{k\geq 0} \dfrac{-(k+1)(k+2)}{4^{k+3}} z^k$, with the same radius of convergence $R = 4$.
Composing with z^5, we have $\dfrac{2}{(z^5-4)^3} = \sum_{k\geq 0} \dfrac{-(k+1)(k+2)}{4^{k+3}} z^{5k}$, with radius of convergence $R = \sqrt[5]{4}$.
Finally, multiplying by $z^2/2$, we have $g(z) = \sum_{k\geq 0} \dfrac{-(k+1)(k+2)}{2 \cdot 4^{k+3}} z^{5k+2}$, with radius of convergence $R = \sqrt[5]{4}$.

---

3. Let $h(z) = (z^2+1)\sin(2z^3)$. Compute the following derivatives: $h^{(15)}(0)$, $h^{(16)}(0)$, and $h^{(17)}(0)$.

Answer. Since $\sin z = \sum_{k\geq 0} \dfrac{(-1)^k}{(2k+1)!} z^{2k+1}$, we have $\sin(2z^3) = \sum_{k\geq 0} \dfrac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+3}$, and hence also

$z^2\sin(2z^3) = \sum_{k\geq 0} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+5}$. Adding gives $h(z) = \sum_{k\geq 0} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+3} + \sum_{k\geq 0} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} z^{6k+5}$.
Note that the first sum uses only powers of z that are $3 \pmod 6$, while the second uses only powers that are $5 \pmod 6$. That is, $h(z) = \sum_{n\geq 0} a_n z^n$, where $a_n = \begin{cases} \frac{(-1)^k 2^{2k+1}}{(2k+1)!} & \text{if } n = 6k+3 \text{ for some } k, \\ \frac{(-1)^k 2^{2k+1}}{(2k+1)!} & \text{if } n = 6k+5 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$

So since $15 = 6k+3$ for $k = 2$, we have $h^{(15)}(0) = 15! a_{15} = \frac{15! \cdot 2^5}{5!}$.
And since 16 is neither 3 or $5 \pmod 6$, we have $h^{(16)}(0) = 15! a_{16} = 0$.
Finally, since $17 = 6k+5$ for $k = 2$, we have $h^{(17)}(0) = 17! a_{17} = \frac{17! \cdot 2^5}{5!}$.

---

4. (a) Find the power series centered at $z = 0$ of these two functions:

$$g(z) = ze^{(z^2)} \qquad \text{and} \qquad h(z) = \cos(2z).$$

(b) Consider the power series $\sum_{k=0}^{\infty} a_k z^k$ centered at $z = 0$ for $f(z) = \frac{ze^{(z^2)}}{\cos(2z)}$.
Use part (a) to compute a_k for each of $k = 0, \ldots, 6$.

(c) For $f(z)$ as in part (b), compute $f^{(5)}(0)$.

(d) What is the radius of convergence of the power series in part (b)?
(Briefly) explain why.

Answers. (a): Since $e^z = \sum_{k\geq 0} \frac{z^k}{k!}$, we have $e^{z^2} = \sum_{k\geq 0} \frac{z^{2k}}{k!}$, and hence $g(z) = ze^{z^2} = \sum_{k\geq 0} \frac{z^{2k+1}}{k!}$.
Since $\cos z = \sum_{k\geq 0} \frac{(-1)^k z^{2k}}{(2k)!}$, we have $h(z) = \cos 2z = \sum_{k\geq 0} \frac{(-4)^k z^{2k}}{(2k)!}$.

(b): Writing each of the power series of part (a) up to the z^6 terms, we have

$$g(z) = z + z^3 + \frac{z^5}{2} + O(z^7) \quad \text{and} \quad h(z) = 1 - 2z^2 + \frac{2}{3}z^4 - \frac{4}{45}z^6 + O(z^7).$$

Thus,

$$\begin{aligned} \frac{1}{h(z)} &= \frac{1}{1 - (2z^2 - \frac{2}{3}z^4 + \frac{4}{45}z^6 + O(z^7))} = 1 + (2z^2 - \frac{2}{3}z^4 + \frac{4}{45}z^6) + (2z^2 - \frac{2}{3}z^4)^2 + (2z^2)^3 + O(z^7) \\ &= 1 + 2z^2 + \Big(-\frac{2}{3} + 4\Big)z^4 + \Big(\frac{4}{45} - \frac{8}{3} + 8\Big)z^6 + O(z^7) = 1 + 2z^2 + \frac{10}{3}z^4 + \frac{244}{45}z^6 + O(z^7) \end{aligned}$$

Multiplying, then, we have

$$f(z) = \frac{g(z)}{h(z)} = z + (1+2)z^3 + \Big(\frac{1}{2} + 2 + \frac{10}{3}\Big)z^5 + O(z^7) = z + 3z^3 + \frac{35}{6}z^5 + O(z^7).$$

That is, $a_0 = a_2 = a_4 = a_6 = 0$, $a_1 = 1$, $a_3 = 3$, and $a_5 = 35/6$.

(c): We have $f^{(5)}(0) = 5! \cdot a_5 = \frac{120 \cdot 35}{6} = 20 \cdot 35 = 700$

(d): $f = g/h$ is analytic everywhere that $h(z) \neq 0$. In particular, the closest points to 0 at which $\cos(2z) = 0$ are where $2z = \pm\pi/2$, i.e., $z = \pm\pi/4$. In particular, f is analytic on $D(0, \pi/4)$ but not on any larger disk centered at 0. Since $h(z) = 0$ at these two points but $g(z)$ is nonzero there, f actually blows up at those points, so there is no function agreeing with f on $D(0, \pi/4)$ but analytic on a larger disk. Thus, the radius of convergence of the power series must be $R = \pi/4$.

---

5. Let $D = \{z \in \mathbb{C} : \operatorname{Re} z \leq -2\}$. Prove that $\displaystyle\sum_{n=1}^{\infty} n^z$ converges uniformly on D, where n^z denotes $e^{z \operatorname{Log} n}$.

Proof. For each integer $n \geq 1$, define $M_n = 1/n^2$. For any $z \in D$ and any $n \geq 1$, writing $z = x + iy$, we have

$$|n^z| = |e^{(x+iy)\log n}| = |e^{x \log n} e^{iy \log n}| = |e^{x \log n}| = n^x \leq n^{-2} = M_n.$$

Meanwhile, $\sum_{n \geq 1} M_n = \sum_{n \geq 1} n^{-2}$ converges by the p-test, since $2 > 1$. Thus, the original sum converges uniformly on D, by the Weierstrass M-Test. QED

---

6. Let $E = \{z \in \mathbb{C} : |z| \geq 7\}$. Prove that $\displaystyle\sum_{k=1}^{\infty} \frac{z^k}{5k - z^{3k}}$ converges uniformly on E.

Proof. For each integer $k \geq 1$, define $M_k = 1/7^k$. For any $z \in E$ and any $k \geq 1$, note that

$$|z|^{3k} - |z|^{2k} \geq |z|^{2k}(|z|^k - 1) \geq 49^k(7 - 1) \geq 5k,$$

and hence $|5k - z^{3k}| \geq |z^{3k}| - |5k| \geq |z|^{2k}$. Thus, $\left|\dfrac{z^k}{5k - z^{3k}}\right| \leq \dfrac{|z|^k}{|z|^{2k}} = \dfrac{1}{|z|^k} \leq \dfrac{1}{7^k} = M_k$. Meanwhile, $\sum_{k \geq 1} M_k = \sum_{k \geq 1} 7^{-k}$ converges by the Geometric Series Test (since $r = 1/7$ has $|r| < 1$). Thus, the original sum converges uniformly on E, by the Weierstrass M-Test. QED

---

7. Let f be an entire function with the property that for all $z \in \mathbb{C}$,

$$f(z + 1) = f(z + i) = f(z).$$

Prove that f is constant.

Proof. Let E be the closed square with vertices at 0, 1, i, $1 + i$; that is, $E = \{x + iy : x, y \in [0, 1]\}$. Since E is closed and bounded, it is compact. Since f is entire, the real-valued function $|f(z)|$ is continuous on E and hence attains a maximum value $M \in \mathbb{R}$. We claim that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Given $z \in \mathbb{C}$, write $z = x + iy$, and write $x = m + s$ and $y = n + t$ with $m, n \in \mathbb{Z}$ and $s, t \in [0, 1)$. By repeated application of the equations of the hypotheses [technically by induction on $m, n \geq 0$ and then by a quick extra argument for $m, n \leq -1$], we have $f(z) = f(s + it)$. Since $s + it \in E$, we have $|f(z)| = |f(s + it)| \leq M$, proving the claim.
Thus, f is both entire and bounded. By Liouville's Theorem, f is constant. QED