Math 345, Fall 2024

Solutions to Homework #9

Problem 1. II.6, #5. Let $0 < B < \pi$. Find a conformal map of the wedge $\{-B < \operatorname{Arg} z < B\}$ onto the right half-plane $\{\operatorname{Re} w > 0\}$. (And of course, verify/prove all your claims.)

Solution. [Inspired by thinking of how z^2 and $\sqrt{z}$ worked.]

Let $f(z) = z^{\pi/2B}$, i.e., $\boxed{f(z) = e^{(\pi/2B)\operatorname{Log} z}}$

Note that $\operatorname{Log} z = \log|z| + i \operatorname{Arg} z$ maps the wedge to the horizontal strip $\{-B < \operatorname{Im} z < B\}$, and then by the previous problem, $w = e^{(\pi/2B)z}$ maps that strip to the right half-plane.

In addition, by Problem 1 of Homework 7 (i.e., II.4 #2), we have $f'(z) = (\pi/2B)f(z)/z$, which is never zero on the wedge, since $f(z)$ is never zero on the wedge. Therefore, by the Theorem on page 59, f provides the desired conformal map.

[Note: There are other correct answers, but this is probably the easiest one to write down.]

Problem 2. II.7, #1(c). Find the fractional linear transformation $f(z) = (az+b)/(cz+d)$ such that

$$f(\infty) = 0, \qquad f(1+i) = 1, \qquad f(2) = \infty$$

Solution. Since $a/c = f(\infty) = 0$, we must have $a = 0$.
So also $c \neq 0$, so cancelling if necessary, we may assume $c = 1$.
Since $b/(2+d) = f(2) = \infty$, we must have $d = -2$. So $f(z) = b/(z-2)$.
Finally, $b/(-1+i) = f(1+i) = 1$, so $b = -1+i$.

That is, $\boxed{f(z) = \dfrac{-1+i}{z-2}}$

Problem 3. II.7, #1(d). Find the fractional linear transformation $f(z) = (az+b)/(cz+d)$ such that

$$f(-2) = 1-2i, \qquad f(i) = 0, \qquad f(2) = 1+2i$$

Solution. Since $f(i) = 0$, the numerator must be $a(z-i)$, so we may assume $a = 1$ and $b = i$.
That is, $f(z) = \dfrac{z-i}{cz+d}$.
Since $\dfrac{-2-i}{-2c+d} = f(-2) = 1-2i$, we have

$$-2c + d = \frac{-2-i}{1-2i} = \frac{(-2-i)(1+2i)}{(1-2i)(1+2i)} = \frac{-2-i-4i+2}{5} = -i.$$

Since $\dfrac{2-i}{2c+d} = f(2) = 1+2i$, we have

$$2c + d = \frac{2-i}{1+2i} = \frac{(2-i)(1-2i)}{(1+2i)(1-2i)} = \frac{2-i-4i-2}{5} = -i.$$

Since $2c + d = -i = -2c + d$, it follows that $c = 0$ and $d = -i$.
That is, $f(z) = \dfrac{z-i}{-i}$, i.e., $\boxed{f(z) = iz + 1}$

Problem 4. II.7, #1(f). Find the fractional linear transformation $f(z) = (az+b)/(cz+d)$ such that

$$f(0) = 0, \qquad f(\infty) = 1, \qquad f(i) = \infty$$

Solution. Since $f(0) = 0$, the numerator must be az, so we may assume $a = 1$ and $b = 0$.
Since $a/c = f(\infty) = 1$, we must have $c = a = 1$. Thus, $f(z) = \dfrac{z}{z+d}$.

Since $f(i) = \infty$, we must have $d = -i$. Thus, $\boxed{f(z) = \dfrac{z}{z-i}}$

Problem 5. II.7, #3. Let $f(z) = (az+b)/(cz+d)$ be the fractional linear transformation such that $f(1) = i$, $f(0) = 1+i$, and $f(-1) = 1$.

Detemine the image of the following three sets under f:

- the unit circle $|z| = 1$.
- the open unit disk $|z| < 1$.
- the imaginary axis $\mathrm{Re}(z) = 0$.

Illustrate with a single sketch showing all three images. (And of course, as on every HW problem, (briefly) explain your reasoning.)

Solution. Since $b/d = f(0) = 1+i$, we may assume that $b = 1+i$ and $d = 1$.

Since $f(1) = i$, we have $(a+(1+i))/(c+1) = i$, so that $a+1+i = ic+i$, and hence $a+1 = ic$.

Since $f(-1) = 1$, we have $(-a+(1+i))/(-c+1) = 1$, so that $-a+1+i = -c+1$, and hence $c = a-i$.

Plugging $c = a-i$ into $a+1 = ic$ yields $a+1 = ia+1$, so that $(1+i)a = 0$, and hence $a = 0$. Thus, $c = 0-i = -i$.

That is, $f(z) = \dfrac{1+i}{1-iz}$.

To determine the image of the unit circle $|z| = 1$, we need only find the image of three points on the circle. The image of the circle is then the unique circle or line that passes through the three image points.

We already have $f(1) = i$ and $f(-1) = 1$. We compute $f(-i) = \dfrac{1+i}{1-i(-i)} = \dfrac{1+i}{0} = \infty$.

Thus (because one point on the circle maps to ∞), the image of the unit circle is the straight line through i and 1. That is, the image of the circle $|z| = 1$ is $\boxed{\text{the set of all } x+iy \text{ with } y = 1-x}$

[*Note*: Alternatively, $f(i) = (1+i)/2$, which is collinear with 1 and i. So of course, we get the same line $y = 1-x$ passing through all three.]

The open unit disk is the whole region inside the unit circle, so its image is one of the two half-planes on one or the other side of the above line $y = 1-x$.

Since 0 lies in the unit disk, and $f(0) = 1+i$ has $y > 1-x$, the image of the unit disk must be $\boxed{\text{the half-plane lying above the line } y = 1-x}$

For the image of the imaginary axis, we were given $f(0) = 1+i$, and we have already computed $f(-i) = \infty$, so that the image of the imaginary axis is some straight line passing through $1+i$.

Since the point ∞ lies on every straight line (and hence on the imaginary axis), we also note that $f(\infty) = 0$. Thus, the image of the imaginary axis is the straight line through $1+i$ and 0, which is the set of $x+iy$ with $y = x$.

[*Note*: Alternatively, again using $f(i) = (1+i)/2$, which is collinear with 0 and $1+i$, we again get the same line $y = x$.]

Here is a graph. The unit circle's image is in blue, the unit disk's image is shaded, and the imaginary axis's image is in red.