Math 345, Fall 2024

Solutions to Homework #8

Problem 1. II.5, #1(a), variant. Prove that $u(x, y) = x^2 - y^2$ is harmonic on $\mathbb{C}$, and find a harmonic conjugate for it.

Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous. We find a harmonic conjugate v by solving the Cauchy-Riemann equations. (Then $u + iv$ is necessarily analytic, so u is harmonic.)

We have $v_y = u_x = 2x$, so $v = 2xy + h(x)$ for some function $h(x)$.

Then $v_x = 2y + h'(x)$, but since $v_x = -u_y = -(-2y) = 2y$, it follows that $h'(x) = 0$. Thus, we may use $h(x) = 0$.

So $\boxed{v(x, y) = 2xy}$ is a harmonic conjugate of u.

The associated analytic function f has $f(x+iy) = x^2 - y^2 + 2xyi = (x+iy)^2$, so $\boxed{f(z) = z^2}$ is analytic with $u = \operatorname{Re} f$. QED

Problem 2. II.5, #1(b), variant. Prove that $u(x, y) = xy + 3x^2y - y^3$ is harmonic on $\mathbb{C}$, and find a harmonic conjugate for it.

Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous. We find a harmonic conjugate v by solving the Cauchy-Riemann equations.

We have $v_y = u_x = y + 6xy$, so $v = \frac{1}{2}y^2 + 3xy^2 + h(x)$ for some function $h(x)$.

Then $v_x = 3y^2 + h'(x)$, but since $v_x = -u_y = -(x + 3x^2 - 3y^2) = -x - 3x^2 + 3y^2$,
it follows that $h'(x) = -x - 3x^2$. Thus, we may use $h(x) = -\frac{1}{2}x^2 - x^3$.

So $\boxed{v(x, y) = \frac{1}{2}(y^2 - x^2) + 3xy^2 - x^3}$ is a harmonic conjugate of u.

The associated analytic function f has

$$f(x + iy) = \frac{1}{2}\big(2xy - i(x^2 - y^2)\big) + \big(-y^3 + 3xy^2 i + 3x^2y - x^3 i\big) = \frac{-i}{2}(x + iy)^2 - i(x + iy)^3.$$

So $\boxed{f(z) = \frac{-i}{2}z^2 - iz^3}$ is analytic with $u = \operatorname{Re} f$. QED

Problem 3. II.5, #1(d), variant. Prove that $u(x, y) = e^{x^2-y^2}\cos(2xy)$ is harmonic on $\mathbb{C}$, and find a harmonic conjugate for it.

Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous. We find a harmonic conjugate v by solving the Cauchy-Riemann equations.

We have $v_y = u_x = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy)$,
so $v = e^{x^2-y^2}\sin(2xy) + h(x)$ for some function $h(x)$.

[**Note**: After taking the derivative u_x, I guessed-and-checked how to do the antiderivative of v_y. You could also use integration by parts, but that would take some time.]

Then $v_x = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy) + h'(x)$,
but since $v_x = -u_y = -(x+3x^2-3y^2) = -\big(-2xe^{x^2-y^2}\sin(2xy) - 2ye^{x^2-y^2}\cos(2xy)\big)$,
it follows that $h'(x) = 0$. Thus, we may use $h(x) = 0$.

So $\boxed{v(x,y) = e^{x^2-y^2}\sin(2xy)}$ is a harmonic conjugate of u.

The associated analytic function f has
$f(x+iy) = e^{x^2-y^2}\big(\cos(2xy) + i\sin(2xy)\big) = e^{(x^2-y^2)+2xyi} = e^{(z^2)}$.

So $\boxed{f(z) = e^{z^2}}$ is analytic with $u = \operatorname{Re} f$. QED

Problem 4. II.5, #1(f), variant. Prove that $u(x,y) = x/(x^2+y^2)$ is harmonic on $\mathbb{C} \setminus \{0\}$, and find a harmonic conjugate for it.
Find a simple formula (in terms of z) for an analytic function $f(z)$ for which $u = \operatorname{Re} f$.

Solution/Proof. Clearly u has all of its partial derivatives continuous on $\mathbb{C} \setminus \{0\}$. We find a harmonic conjugate v by solving the Cauchy-Riemann equations.

We have $u_x = \dfrac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2-x^2}{(x^2+y^2)^2}$.
Seeing how that derivative worked out, it's easy to guess-and-check that with $v_y = u_x$, we have
$v = \dfrac{-y}{x^2+y^2} + h(x)$ for some function $h(x)$.

Then $v_x = \dfrac{2xy}{(x^2+y^2)^2} + h'(x)$,

but since $v_x = -u_y = \dfrac{2xy}{(x^2+y^2)^2}$, it follows that $h'(x) = 0$. Thus, we may use $h(x) = 0$.

So $\boxed{v(x,y) = \dfrac{-y}{x^2+y^2}}$ is a harmonic conjugate of u.

The associated analytic function f has
$f(x+iy) = \dfrac{x-iy}{x^2+y^2} = \dfrac{\overline{z}}{|z|^2} = \dfrac{\overline{z}}{z\overline{z}} = \dfrac{1}{z}$.

So $\boxed{f(z) = \dfrac{1}{z}}$ is analytic with $u = \operatorname{Re} f$. QED

Problem 5. II.6, #4. Let $A > 0$ be a positive real number. Find a conformal map of the horizontal strip $\{-A < \operatorname{Im} z < A\}$ onto the right half-plane $\{\operatorname{Re} w > 0\}$. (And of course, verify/prove all your claims.)

Solution. Recall that e^z maps the horizontal strip $\{-\pi/2 < \operatorname{Im} z < \pi/2\}$ to the right half-plane, since $\operatorname{Arg}(e^z) = \operatorname{Im} z$, and $|e^z| = e^{\operatorname{Re} z}$.

And of course, multiplication by $\pi/(2A)$ maps the strip $\{-A < \operatorname{Im} z < A\}$
onto the strip $\{-\pi/2 < \operatorname{Im} z < \pi/2\}$.

Composing these two maps, then, the analytic function $\boxed{f(z) = e^{(\pi/2A)z}}$
maps the strip $\{-A < \operatorname{Im} z < A\}$ to the half-plane $\{\operatorname{Re} w > 0\}$.

In addition, $f'(z) = \dfrac{\pi}{2A}e^{(\pi/2A)z}$ is never 0. Therefore, by the Theorem on page 59, f provides the desired conformal map.

[Note: There are other correct answers, but this is probably the easiest one to write down.]