Math 345, Fall 2024

Solutions to Homework #7

Problem 1. II.4, #2. Let $a \in \mathbb{C} \smallsetminus \{0\}$ be a nonzero complex number, and Let $f(z)$ be an analytic branch of z^a . Prove that $f'(z) = af(z)/z$.

Proof. We have $f(z) = \exp\big(ag(z)\big)$ where $\exp(z) = e^z$, and $g(z)$ is some (analytic) branch of $\log z$.

Note that the domain D of g is also the domain of f. For any $z \in D$, we have $g'(z) = \dfrac{1}{z}$. (See the sentence just after equation (4.2) on page 52.)

Therefore, since $\dfrac{d}{dz}(\exp(z)) = \exp(z)$, at any $z \in D$, the Chain Rule gives us

$f'(z) = \exp\big(ag(z)\big) \cdot ag'(z) = f(z) \cdot a \cdot \dfrac{1}{z} = \dfrac{af(z)}{z}$ QED

Problem 2. II.4, #7. Let $D \subseteq \mathbb{C}$ be a bounded domain, and let f be a bounded analytic function on D. Suppose also that f is one-to-one on D. Prove that $\text{Area}\big(f(D)\big) = \iint_D \big|f'(z)\big|^2 \, dx\, dy$.

Proof. From the Theorem on page 51, the Jacobian of f, viewed as a map from D to $\mathbb{R}^2$, satisfies $|\det J_f| = |f'(z)^2|$. Therefore, by the Multivariable Change-of-Variables Formula,

$\text{Area}\big(f(D)\big) = \iint_{f(D)} 1\, du\, dv = \iint_D 1 \cdot \big|f'(z)\big|^2 \, dx\, dy = \iint_D \big|f'(z)\big|^2 \, dx\, dy.$ QED

Problem 3. II.4, #9. Let $D = D(0,1)$ be the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$, and let $f(z) = z^2$. Compute $\iint_D \big|f'(z)\big|^2 \, dx\, dy$. Interpret the answer in terms of areas; that is, explain how the value you get actually agrees with the previous problem.

Solution. We have $f'(z) = 2z$, so $|f'(z)|^2 = 4|z|^2 = 4(x^2+y^2)$. We now have a Math 211-style double integral, which we compute with polar coordinates:

$$\iint_D \big|f'(z)\big|^2 \, dx\, dy = \iint_D 4(x^2+y^2)\, dx\, dy = \int_0^{2\pi}\int_0^1 4r^2 \cdot r\, dr\, d\theta = \int_0^{2\pi}\int_0^1 4r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} r^4\Big|_0^1 \, d\theta = \int_0^{2\pi} (1-0)\, d\theta = \theta\Big|_0^{2\pi} = \boxed{2\pi}$$

Interpretation. The image disk $f(D)$ is the same disk $D(0,1)$, which has area π, not 2π. But of course, the function $f : D \to f(D)$ is not one-to-one, but rather two-to-one, so the previous problem does not apply directly. Instead, since the mapping is two-to-one, we get 2 times the area as the value of the integral.

More precisely, let's cut D into two pieces, the right half $U_1 = \{z \in D : \operatorname{Re} z > 0\}$ and the left half $U_2 = \{z \in D : \operatorname{Re} z < 0\}$. Then each half maps one-to-one onto $D(0,1)$ (well, aside from removing a slit of area 0). So the contribution to the integral above from each of U_1 and U_2 alone is π, for the area π of its image $D(0,1)$ (minus the slit), totaling 2π.

Problem 4. II.5, #2. Let $D \subseteq \mathbb{C}$ be a domain, and let $u, v : D \to \mathbb{R}$ be harmonic functions. Suppose that v is a harmonic conjugate for u. Prove that $-u$ is a harmonic conjugate for v.

Proof. Let $f = u + iv$, which is analytic on D because v is a harmonic conjugate for u. Let $g = -if = v - iu$. Then (as a constant multiple of f), g is also analytic. Since $v = \operatorname{Re} g$, then by definition of harmonic conjugate, we have that $-u = \operatorname{Im} g$ is a harmonic conjugate for v. QED

Problem 5. II.5, #3(a,b), slight variant. Define $u(z) = \begin{cases} \mathrm{Im}(1/z^2) & \text{for } z \in \mathbb{C} \setminus \{0\}, \\ 0 & \text{for } z = 0. \end{cases}$

Prove that all four of $\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial u}{\partial y}, \dfrac{\partial^2 u}{\partial y^2}$ exist at *all* points of $\mathbb{C}$ (viewed as $\mathbb{R}^2$), including at the point $(0,0)$.

Then verify that u satisfies Laplace's equation on $\mathbb{R}^2$. That is, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ everywhere on $\mathbb{R}^2$.

Proof. Since $u(z) = \mathrm{Im}(\bar{z}^2/|z|^4)$ for $z \neq 0$, a quick computation shows that $u(x,y) = \dfrac{-2xy}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$.

Away from $(0,0)$, $u(x,y)$ is a rational function defined everywhere on $\mathbb{R}^2 \setminus \{(0,0)\}$, so (by the quotient rule, etc.), its partial derivatives u_x and u_y are also rational functions defined everywhere on $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus, *their* partial derivatives u_{xx} and u_{yy} are also defined everywhere on $\mathbb{R}^2 \setminus \{(0,0)\}$. [Note: you can also compute their formulas exactly, but we won't need to here.]

Note that $u(x,0) = 0$ for all $x \in \mathbb{R}$, and $u(0,y) = 0$ for all $y \in \mathbb{R}$. Thus, by definition of partial derivatives, for every $x, y \in \mathbb{R}$, we have

$$u_x(x,0) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, 0) - u(x,0)}{\Delta x} = \lim_{\Delta x \to 0} 0 = 0$$

and

$$u_y(0,y) = \lim_{\Delta y \to 0} \frac{u(0, y+\Delta y) - u(0,y)}{\Delta y} = \lim_{\Delta y \to 0} 0 = 0.$$

In particular, both $u_x(0,0)$ and $u_y(0,0)$ exist and equal 0.

Since $u_x(x,0)$ is identically zero on the x-axis, it follows that $u_{xx}(x,0)$ is also identically zero. In particular, $u_{xx}(0,0)$ exists and equals 0. Similarly, $u_{yy}(0,y)$ is identically zero, and in particular, $u_{yy}(0,0)$ exists and equals 0. We have shown that u_x, u_y, u_{xx}, and u_{yy} are defined at *all* points of $\mathbb{R}^2$, as desired.

For Laplace's equation, note that at $(0,0)$, we have just shown that $u_{xx}(0,0) = u_{yy}(0,0) = 0$, and hence $u_{xx}(0,0) + u_{yy}(0,0) = 0$. Away from the origin, $u = \mathrm{Im}(1/z^2)$ is the imaginary part of an analytic function (namely $f(z) = 1/z^2$) and hence is harmonic on $\mathbb{C} \setminus \{0\}$ as well. [This can also be confirmed by computing u_{xx} and u_{yy} directly, but that's much more painful.] Thus, u satisfies Laplace's equation on all of $\mathbb{R}^2$. QED

Side note: If you really want to compute derivatives, then away from $(0,0)$, direct computation gives

$$u_x(x,y) = \frac{-2y(x^2+y^2)^2 + 2xy \cdot 2(x^2+y^2)(2x)}{(x^2+y^2)^4} = \frac{2y(-x^2-y^2+4x^2)}{(x^2+y^2)^3} = \frac{2y(3x^2-y^2)}{(x^2+y^2)^3}$$

and similarly $u_y = \dfrac{2x(3y^2-x^2)}{(x^2+y^2)^3}$.

Another pair of brute-force computations then gives $u_{xx} = \dfrac{24xy(y^2-x^2)}{(x^2+y^2)^4}$ and $u_{yy} = \dfrac{24xy(x^2-y^2)}{(x^2+y^2)^4}$ away from $(0,0)$.

And those two expressions for u_{xx} and u_{yy} clearly sum to 0 for $(x,y) \neq (0,0)$. But even if you do all of these explicit computations, you still need to go back and compute all of these partial derivatives at $(0,0)$ by the limit definition, as in the main proof above.

Problem 6. II.5, #3(c,d), slight variant. With u as in the previous problem, prove that $\dfrac{\partial^2 u}{\partial x \partial y}$ does *not* exist at $(0,0)$. Conclude that u is *not* harmonic on the whole plane, even though we just saw that it satisfies Laplace's equation on the whole plane.

Proof. This time, we *do* need to compute at least u_y explicitly away from $(0,0)$, giving:

$$u_y(x,y) = \frac{-2x(x^2+y^2)^2 + 2xy \cdot 2(x^2+y^2)(2y)}{(x^2+y^2)^4} = \frac{2y(-x^2-y^2+4y^2)}{(x^2+y^2)^3} = \frac{2y(3y^2-x^2)}{(x^2+y^2)^3}$$

As we saw in the previous problem, we also have $u_y(0,0) = 0$. Thus,

$$u_y(x,y) = \begin{cases} \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0), \end{cases}$$

Restricting to $y = 0$, we have

$$u_y(x,0) = \begin{cases} -2x^{-3} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

which is not continuous and hence not differentiable at $x = 0$, since $\lim_{x \to 0} \dfrac{-2}{x^3}$ (for $x \in \mathbb{R}$) diverges. Thus, $u_{yx}(0,0)$ does not exist, as desired.

In addition, it follows that u does *not* have all of its second partials existing, so u is *not* harmonic on the whole plane. QED

Side note: Actually, the original function u isn't even *continuous* on the plane.

Indeed, for any $x \in \mathbb{R} \setminus \{0\}$, we have $u(x,x) = \dfrac{-2x^2}{(x^2+x^2)^2} = -\dfrac{1}{2x^2}$.

Thus, $\lim_{x \to 0} u(x,x) = \lim_{x \to 0} -\dfrac{1}{2x^2} = -\infty$ diverges, so indeed, u is not even continuous at $(0,0)$.