Math 345, Fall 2024

Solutions to Homework #6

Problem 1. II.1, #16, slight variant: Prove that
(a) the slit plane $\mathbb{C} \setminus (-\infty, 0]$ is star-shaped, but
(b) the punctured plane $\mathbb{C} \setminus \{0\}$ is *not* star-shaped.

Proof. (a) Let $D = \mathbb{C} \setminus (-\infty, 0]$. We will show that $z_0 = 1 \in D$ works as the star point of this domain.

Given any $z \in D$, write $z = x + iy$. To prove that the line segment L from 1 to z is contained in D, we consider two cases.

Case 1. Suppose $y = 0$. Then $z \in \mathbb{R}$, and hence we must have $z > 0$. Thus, the line segment L is $[1, z]$ if $z \geq 1$, or $[z, 1]$ if $0 < z < 1$. Either way, L is contained in the positive real line $(0, \infty)$, and hence it is contained in D.

Case 2. Otherwise, we have $y \neq 0$. Then every point w on L besides the endpoint $z_0 = 1$ has nonzero imaginary part, and hence $w \in D$. Since $1 \in D$ as well, it follows that $L \subseteq D$. QED (a)

(b): Let $D' = \mathbb{C} \setminus \{0\}$. Given any $z_0 \in D'$, we must show that z_0 does not serve as a star point of D'.

Let $z = -z_0$. Then $z \neq 0$, since $z_0 \neq 0$; that is, $z \in D'$. However, the line segment L from z_0 to z contains the point $0 \notin D'$. Thus, $L \not\subseteq D'$, proving that z_0 is not a star point of D'. QED (b)

Problem 2. II.2, #2. For any $z \in \mathbb{C} \setminus \{1\}$ and any integer $n \geq 1$, prove that

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} = \frac{1 - z^n}{(1 - z)^2} - \frac{nz^n}{1 - z}.$$

Proof. **(Method 1)**: By induction on $n \geq 1$.

For $n = 1$, the left side is 1, and the right side is $\frac{1 - z}{(1 - z)^2} - \frac{z}{1 - z} = \frac{1}{1 - z} - \frac{z}{1 - z} = \frac{1 - z}{1 - z} = 1$, proving the base case.

Assuming the equality holds for some particular $n \geq 1$, we must show it also holds for $n + 1$. We have

$$1 + 2z + 3z^2 + \cdots + nz^{n-1} + (n+1)z^n = \left(\frac{1 - z^n}{(1 - z)^2} - \frac{nz^n}{1 - z}\right) + (n+1)z^n \quad \text{[by the inductive assumption]}$$

$$= \frac{1 - z^n}{(1 - z)^2} + \left(-\frac{nz^n}{1 - z} + (n+1)z^n\right) = \frac{1 - z^{n+1} + z^{n+1} - z^n}{(1 - z)^2} + \frac{(-nz^n) + (n+1)z^n - (n+1)z^{n+1}}{1 - z}$$

$$= \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{z^n(1 - z)}{(1 - z)^2} + \frac{z^n - (n+1)z^{n+1}}{1 - z} = \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{z^n}{1 - z} + \frac{z^n}{1 - z} - \frac{(n+1)z^{n+1}}{1 - z}$$

$$= \frac{1 - z^{n+1}}{(1 - z)^2} - \frac{(n+1)z^{n+1}}{1 - z}$$

QED

(Method 2): For any such z and n, we have

$$(1 - z)\big(1 + z + z^2 + \cdots + z^{n-1}\big) = \big(1 + z + z^2 + \cdots + z^{n-1}\big) - \big(z + z^2 + z^3 + \cdots + z^n\big) = 1 - z^n.$$

Therefore, we have

$$(1 - z)^2\big(1 + 2z + 3z^2 + \cdots + nz^{n-1}\big) = (1 - z)\Big[\big(1 + 2z + 3z^2 + \cdots + nz^{n-1}\big) - \big(z + 2z^2 + 3z^3 + \cdots + nz^n\big)\Big]$$

$$= (1-z)\Big[\big(1+z+z^2+\cdots+z^{n-1}\big) - nz^n\Big] = (1-z^n) - nz^n(1-z).$$

Dividing both sides by $(1-z)^2$, then, we have $1+2z+3z^2+\cdots+nz^{n-1} = \dfrac{1-z^n}{(1-z)^2} - \dfrac{nz^n}{1-z}$. QED

——— ——— ——— ——— ——— ——— ——— ———

(Method 3): Start the same way as Method 2, to get:

$(1-z)\big(1+z+z^2+\cdots+z^n\big) = 1-z^{n+1}$, and hence $1+z+z^2+\cdots+z^n = \dfrac{1-z^{n+1}}{1-z}$.

Both sides are analytic on $\mathbb{C} \smallsetminus \{1\}$, so we may differentiate, to get:

$$1+2z+3z^2+\cdots+nz^{n-1} = \frac{-(n+1)z^n(1-z) - (1-z^{n+1})(-1)}{(1-z)^2}$$

$$= \frac{-nz^n(1-z) - z^n + z^{n+1} + 1 - z^{n+1}}{(1-z)^2} = \frac{-nz^n(1-z) + 1 - z^n}{(1-z)^2} = \frac{1-z^n}{(1-z)^2} - \frac{nz^n}{1-z}$$

QED

Problem 3. II.2, #3. Prove (from the definition) that the functions $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are not differentiable at any point.

Proof. For any $z \in \mathbb{C}$, the limit definition of the derivative of $\mathrm{Re}(z)$ is

$$\mathrm{Re}'(z) = \lim_{h\to 0} \frac{\mathrm{Re}(z+h) - \mathrm{Re}(z)}{h} = \lim_{h\to 0} \frac{\mathrm{Re}\, h}{h}.$$

Writing $h = u + iv$, first let $v = 0$, so that $h = u$. Then the limit above (along the horizontal line $h \in \mathbb{R}$) is

$$\lim_{u\to 0} \frac{\mathrm{Re}(u+0i)}{u+0i} = \lim_{u\to 0} \frac{u}{u} = \lim_{u\to 0} 1 = 1$$

But if we use $u = 0$, so that $h = iv$, then the limit above (along the vertical line $h \in i\mathbb{R}$) is

$$\lim_{v\to 0} \frac{\mathrm{Re}(0+iv)}{0+iv} = \lim_{v\to 0} \frac{0}{iv} = \lim_{v\to 0} 0 = 0.$$

Since these two limits disagree, the overall limit defining $\mathrm{Re}'(z)$ diverges. QED Re

——— ——— ——— ——— ——— ——— ——— ———

For any $z \in \mathbb{C}$, the limit definition of the derivative of $\mathrm{Im}(z)$ is

$$\mathrm{Im}'(z) = \lim_{h\to 0} \frac{\mathrm{Im}(z+h) - \mathrm{Im}(z)}{h} = \lim_{h\to 0} \frac{\mathrm{Im}\, h}{h}.$$

Writing $h = u + iv$, first let $v = 0$, so that $h = u$. Then the limit above (along the horizontal line $h \in \mathbb{R}$) is

$$\lim_{u\to 0} \frac{\mathrm{Im}(u+0i)}{u+0i} = \lim_{u\to 0} \frac{0}{u} = \lim_{u\to 0} 0 = 0$$

But if we use $u = 0$, so that $h = iv$, then the limit above (along the vertical line $h \in i\mathbb{R}$) is

$$\lim_{v\to 0} \frac{\mathrm{Im}(0+iv)}{0+iv} = \lim_{v\to 0} \frac{v}{iv} = \lim_{v\to 0} -i = -i.$$

Since these two limits disagree, the overall limit defining $\mathrm{Im}'(z)$ diverges. QED Im

---

Problem 4. II.3, #2. Prove that the functions $u = \sin x \sinh y$ and $v = \cos x \cosh y$ satisfy the Cauchy-Riemann equations. Then find a function $f(z)$ (with a simple formula in terms of z) so that $f = u + iv$. (And of course, prove/verify that this formula holds.)

Proof. We compute $\dfrac{\partial u}{\partial x} = \cos x \sinh y$ and $\dfrac{\partial v}{\partial y} = \cos x \sinh y = \dfrac{\partial u}{\partial x}$.

Similarly, We compute $\dfrac{\partial u}{\partial y} = \sin x \cosh y$ and $\dfrac{\partial v}{\partial x} = -\sin x \cosh y = -\dfrac{\partial u}{\partial y}$. QED

——— ——— ——— ——— ——— ——— ——— ———

By I.8 #1(a) [from HW3], we have $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Thus, using the formulas $\cos(iz) = \cosh(z)$ and $\sin(iz) = i\sinh(z)$ from Section I.8 (page 30), we have $\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$.

Therefore, $f = u + iv = \sin x \sinh y + i \cos x \cosh y = i\cos(x+iy) = i\cos(z)$. So $\boxed{f(z) = i\cos(z)}$

Problem 5. II.3, #3. Let $D \subseteq \mathbb{C}$ be a domain and let $f : D \to \mathbb{C}$. Suppose that both $f(z)$ and its complex conjugate $\overline{f}(z)$ are analytic on D. Prove that f is constant on D.

Proof. Write $f = u + iv$, so that $\overline{f} = u - iv$. Since both are analytic, we have $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x}$, by the first Cauchy-Riemann equation for f and for $\overline{f}$, respectively. Thus, $\dfrac{\partial u}{\partial x} = 0$ on D.

Similarly, the second Cauchy-Riemann equations for f and for $\overline{f}$ give us $-\dfrac{\partial v}{\partial x} = \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$.

Thus, $\dfrac{\partial v}{\partial x} = 0$ on D.

By equation (3.1), then, we have $f' = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = 0$ on D. By the Theorem on page 49, it follows that f is constant on D. QED