Math 345, Fall 2024

Solutions to Homework #5

Problem 1. Let $\{a_n\} \subseteq \mathbb{C}$ be a sequences. Prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} |a_n| = 0$

Proof. ($\Rightarrow$) Given $\varepsilon > 0$, there is some $N \geq 1$ such that for all $n \geq N$, we have $|a_n - 0| < \varepsilon$.
Given $n \geq N$, we have $\big||a_n| - 0\big| = |a_n| = |a_n - 0| < \varepsilon$ QED ($\Rightarrow$)

($\Leftarrow$) Given $\varepsilon > 0$, there is some $N \geq 1$ such that for all $n \geq N$, we have $\big||a_n| - 0\big| < \varepsilon$.
Given $n \geq N$, we have $|a_n - 0| = |a_n| = \big||a_n| - 0\big| < \varepsilon$ QED

Problem 2. II.1, #1(c): Let $p > 1$. Prove that $\lim_{n\to\infty} \frac{2n^p + 5n + 1}{n^p + 3n + 1} = 2$.

Proof. Multiplying top and bottom by n^{-p} gives

$$\lim_{n\to\infty} \frac{2n^p + 5n + 1}{n^p + 3n + 1} = \lim_{n\to\infty} \frac{2 + 5n^{-(p-1)} + n^{-p}}{1 + 3n^{-(p-1)} + n^{-p}} = \frac{2+0+0}{1+0+0} = 2, \text{ as desired.}$$

Here, we have used the fact that $\lim_{n\to\infty} n^{-r} = 0$ for $r > 0$ (i.e., equation (1.1) page 34), as well as the arithmetic laws for limits (i.e., the Theorem on page 34). QED

Problem 3. II.1, #1(d): Let $z \in \mathbb{C}$. Prove that $\lim_{n\to\infty} \frac{z^n}{n!} = 0$.

Proof. We claim that $\lim_{n\to\infty} \frac{|z|^n}{n!} = 0$.

To prove this claim, let N be a positive integer with $N > |z|$, and let $M = \frac{|z|^N}{N!} \geq 0$. Then for any $n \geq N$, we have

$$0 \leq \frac{|z|^n}{n!} = M \prod_{k=N+1}^{n} \frac{|z|}{k} \leq M\left(\frac{|z|}{N}\right)^{n-N} = Mr^{n-N},$$

where $r = \frac{|z|}{N}$. Since $0 \leq r < 1$, we have

$$\lim_{n\to\infty} Mr^{n-N} = M \lim_{n\to\infty} r^{n-N} = M \cdot 0 = 0.$$

In addition, we have $\lim_{n\to\infty} 0 = 0$. Combining these two limits with the bound above that $0 \leq \frac{|z|^n}{n!} \leq Mr^{n-N}$, the Squeeze Law (or In-Between Theorem) gives us $\lim_{n\to\infty} \frac{|z|^n}{n!} = 0$, proving our claim.

Finally, since $\left|\frac{z^n}{n!}\right| = \frac{|z|^n}{n!}$, the desired result is immediate from Problem 1.

Problem 4. II.1, #7: Define a sequence $\{x_0\}_{n\geq 0} \subseteq \mathbb{R}$ inductively by $x_0 = 0$, and $x_{n+1} = x_n^2 + \frac{1}{4}$ for each $n \geq 0$. Prove that $\lim_{n\to\infty} x_n = \frac{1}{2}$.

Proof. First, we claim that $\{x_n\}$ is increasing. That is, given any $n \geq 0$, we must show that $x_{n+1} \geq x_n$. We have

$$x_{n+1} - x_n = x_n^2 + \frac{1}{4} - x_n = \left(x_n - \frac{1}{2}\right)^2 \geq 0,$$

since $x_n \in \mathbb{R}$. Thus, $x_{n+1} \geq x_n$, as claimed.

Second, we claim that $x_n \leq \dfrac{1}{2}$ for every $n \geq 0$. We prove this by induction on $n \geq 0$. For $n = 0$, we have $x_0 = 0 \leq \dfrac{1}{2}$, as desired.

Now assume $x_n \leq \dfrac{1}{2}$ for some particular $n \geq 0$; we will show the bound for $n+1$. We have

$$\frac{1}{2} - x_{n+1} = \frac{1}{2} - \left(x_n^2 + \frac{1}{4}\right) = \frac{1}{4} - x_n^2 = \left(\frac{1}{2} + x_n\right)\left(\frac{1}{2} - x_n\right) \geq \frac{1}{2} \cdot 0 = 0.$$

Here, we used the fact that $x_n \geq 0$ by our first claim, as well as the inductive hypothesis that $x_n \leq \dfrac{1}{2}$. Thus, $x_{n+1} \leq \dfrac{1}{2}$, proving our second claim.

Thus, $\{x_n\}$ is a bounded, increasing sequence. By the Monotone Sequence Theorem, it converges to some real number $L \in \mathbb{R}$. Therefore,

$$L^2 + \frac{1}{4} = \left(\lim_{n\to\infty} x_n\right)^2 + \frac{1}{4} = \lim_{n\to\infty}\left(x_n^2 + \frac{1}{4}\right) = \lim_{n\to\infty} x_{n+1} = L.$$

Rearranging, the real number L satisfies $L^2 - L + \dfrac{1}{4} = 0$, and hence $\left(L - \dfrac{1}{2}\right)^2 = 0$.

Thus, $L - \dfrac{1}{2} = 0$, and hence $L = \dfrac{1}{2}$. QED

Problem 5. Prove that $\mathbb{R}$ is a closed but not open subset of $\mathbb{C}$.

Proof. (**Closed**): We must prove that $\mathbb{C} \smallsetminus \mathbb{R}$ is open. Given $z_0 \in \mathbb{C} \smallsetminus \mathbb{R}$, let $y_0 = \operatorname{Im} z_0$, which is a nonzero real number. Let $r = |y_0| > 0$. We claim that the disk $D(z_0, y)$ is contained in $\mathbb{C} \smallsetminus \mathbb{R}$.

To prove this claim, given $z \in D(z_0, y)$, we have

$$r = \left|\operatorname{Im}(z_0)\right| = \left|\operatorname{Im}(z) - \operatorname{Im}(z - z_0)\right| \leq \left|\operatorname{Im}(z)\right| + \left|\operatorname{Im}(z - z_0)\right| \leq \left|\operatorname{Im}(z)\right| + \left|z - z_0\right| < \left|\operatorname{Im}(z)\right| + r.$$

Thus, $\left|\operatorname{Im}(z)\right| > 0$, so that $\operatorname{Im} z \neq 0$, and hence $z \notin \mathbb{R}$. That is, $z \in \mathbb{C} \smallsetminus \mathbb{R}$, proving the claim

Hence, we have proven that $\mathbb{C} \smallsetminus \mathbb{R}$. Equivalently, $\mathbb{R}$ is closed. QED Closed

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(**Not Open**): We will show that there is no open disk centered at $0 \in \mathbb{R}$ that is contained in $\mathbb{R}$.

Given $r > 0$, let $z = \dfrac{ri}{2}$. Then $z \in \mathbb{C} \smallsetminus \mathbb{R}$. In addition, $|z - 0| = |z| = \dfrac{r}{2} < r$, proving that $z \in D(0, r)$. So $D(0, r) \not\subseteq \mathbb{C} \smallsetminus \mathbb{R}$. QED Not Open