Math 345, Fall 2024

Solutions to Homework #4

Problems 1–2. Fix $c \in \mathbb{C}$. Prove that the constant function $f(z) = c$ and the identity function $g(z) = z$ are continuous on all of $\mathbb{C}$.

Proof. $\boxed{\text{For } f(z) = c}$ Given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, let $\delta = 1 > 0$.
Given $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta$, we have $|f(z) - f(z_0)| = |c - c| = 0 < \varepsilon$ QED

$\boxed{\text{For } g(z) = z}$ Given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, let $\delta = \varepsilon > 0$.
Given $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta$, we have $|g(z) - g(z_0)| = |z - z_0| < \delta = \varepsilon$ QED

Problems 3–6. Prove that each of the four functions

$$z \mapsto \operatorname{Re} z, \qquad z \mapsto \operatorname{Im} z, \qquad z \mapsto |z|, \qquad z \mapsto \overline{z}$$

is continuous on all of $\mathbb{C}$.

Proof. $\boxed{\text{For } \operatorname{Re} z}$ Given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, let $\delta = \varepsilon > 0$.
Given $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta$, we have
$|\operatorname{Re}(z) - \operatorname{Re}(z_0)| = |\operatorname{Re}(z - z_0)| \le |z - z_0| < \delta = \varepsilon$ QED

$\boxed{\text{For } \operatorname{Im} z}$ Given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, let $\delta = \varepsilon > 0$.
Given $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta$, we have
$|\operatorname{Im}(z) - \operatorname{Im}(z_0)| = |\operatorname{Im}(z - z_0)| \le |z - z_0| < \delta = \varepsilon$ QED

$\boxed{\text{For } |z|}$ Given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, let $\delta = \varepsilon > 0$.
Given $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta$,
observe that $|z| = |(z - z_0) + z_0| \le |z - z_0| + |z_0|$, and hence $|z| - |z_0| \le |z - z_0|$.
Similarly, $|z_0| - |z| \le |z - z_0|$. Since $|z|$ and $|z_0|$ are real numbers, we have $\big||z| - |z_0|\big| \le |z - z_0|$.
Thus, $\big||z| - |z_0|\big| \le |z - z_0| < \delta = \varepsilon$ QED

$\boxed{\text{For } \overline{z}}$ Given $z_0 \in \mathbb{C}$ and $\varepsilon > 0$, let $\delta = \varepsilon > 0$.
Given $z \in \mathbb{C}$ with $0 < |z - z_0| < \delta$, we have
$\left|\overline{z} - \overline{z_0}\right| = \left|\overline{z - z_0}\right| = |z - z_0| < \delta = \varepsilon$ QED

Problem 7. Let $D \subseteq \mathbb{C}$, let $f, g : D \to \mathbb{C}$, let $z_0 \in D$, let $L, M \in \mathbb{C}$, and suppose that $\lim_{z \to z_0} f(z) = L$ and $\lim_{z \to z_0} g(z) = M$. Prove that $\lim_{z \to z_0} f(z) \cdot g(z) = LM$.

Proof. Since $\lim_{z \to z_0} g(z) = M$, there exists $\gamma > 0$ such that for any $z \in D$ with $0 < |z - z_0| < \gamma$, we have $|g(z) - M| < 1$.
Thus, for any such z, we have $|g(z)| \le |g(z) - M| + |M| < 1 + |M|$.
[Note, in particular, that $1 + |M| > 0$, so we may divide by $1 + |M|$ later. Similarly for $1 + |L|$.]

Given $\varepsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that:

For any $z \in D$ with $0 < |z - z_0| < \delta_1$, we have $|f(z) - L| < \dfrac{\varepsilon}{2(1 + |M|)}$, and

For any $z \in D$ with $0 < |z - z_0| < \delta_2$, we have $|g(z) - M| < \dfrac{\varepsilon}{2(1 + |L|)}$.

Let $\delta = \min\{\gamma, \delta_1, \delta_2\} > 0$. Given $z \in D$ with $0 < |z - z_0| < \delta$, we have

$$\big|f(z)g(z) - LM\big| \leq |f(z)g(z) - Lg(z)| + |Lg(z) - LM| = \big|f(z) - L\big| \cdot |g(z)| + |L| \cdot \big|g(z) - M\big|$$

$$< \frac{\varepsilon}{2(1 + |M|)} \cdot (1 + |M|) + |L| \cdot \frac{\varepsilon}{2(1 + |L|)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$ QED

Problems 8–9. Let $a \in \mathbb{C}$ and $r > 0$. Prove that the open disk $D(a, r)$ is indeed an open set, and that the closed disk $\overline{D}(a, r)$ is indeed a closed set.

Proof. $\boxed{D(a, r) \text{ is open}}$ Given $z_0 \in D(a, r)$, we have $|z_0 - a| < r$. Define $s = r - |z_0 - a| > 0$. We claim that $D(z_0, s) \subseteq D(a, r)$.

Given $z \in D(z_0, s)$, we have $|z - a| \leq |z - z_0| + |z_0 - a| < s + |z_0 - a| = r$, and hence $z \in D(a, r)$.

Thus, we have proven our claim and hence proven that $D(a, r)$ is open. QED

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$\boxed{\overline{D}(a, r) \text{ closed}}$ We must show that $\mathbb{C} \setminus \overline{D}(a, r)$ is open.

Given $z_0 \in \mathbb{C} \setminus \overline{D}(a, r)$, we have $|z_0 - a| > r$. Define $s = |z_0 - a| - r > 0$.
We claim that $D(z_0, s) \subseteq \mathbb{C} \setminus \overline{D}(a, r)$.

Given $z \in D(z_0, s)$, we have $|z_0 - a| \leq |z_0 - z| + |z - a|$, and hence

$|z - a| \geq |z_0 - a| - |z_0 - z| > |z_0 - a| - s = r$.

Therefore, $z \in \mathbb{C} \setminus \overline{D}(a, r)$.

Thus, we have proven our claim and hence proven that $\mathbb{C} \setminus \overline{D}(a, r)$ is open.
That is, $\overline{D}(a, r)$ is closed. QED