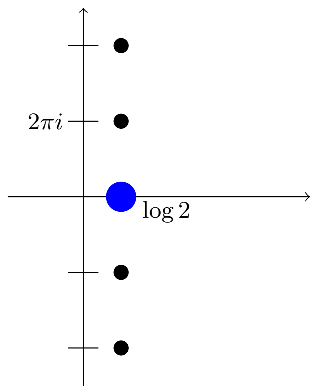
Math 345, Fall 2024

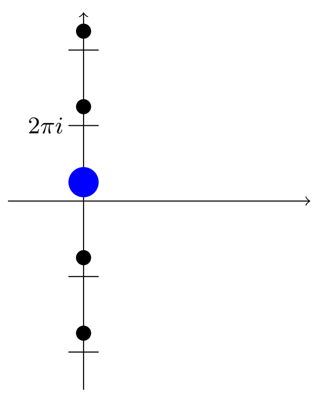
Solutions to Homework #3

I.6, #1(a,b,c,d) Find and plot (all values of) $\log z$, specifying the principal value $\operatorname{Log} z$. (a): 2. (b): i. (c): $1+i$. (d): $(1+i\sqrt{3})/2$.

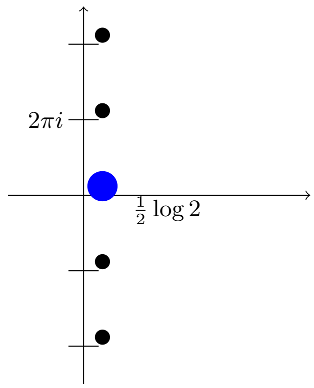
Solutions. (a) We have $|2| = 2$ and $\operatorname{Arg} 2 = 0$, so that $\arg 2 = 0 + 2\pi\mathbb{Z}$.
Therefore, with $z = 2$, $\boxed{\log z = \log 2 + 2\pi n i}$ for all integers $n \in \mathbb{Z}$
The principal value is $\boxed{\log 2}$ (i.e., for $n = 0$). Here's the plot (not to scale):



(b) We have $|i| = 1$ and $\operatorname{Arg} i = \pi/2$, so that $\arg i = \pi/2 + 2\pi\mathbb{Z}$.
Therefore, with $z = i$, $\boxed{\log z = (\pi/2 + 2\pi n)i}$ for all integers $n \in \mathbb{Z}$
The principal value is $\boxed{\pi i/2}$ (i.e., for $n = 0$). Here's the plot (not to scale):

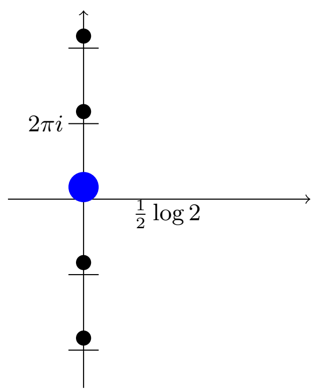


(c) We have $|1+i| = \sqrt{2}$ and $\operatorname{Arg}(1+i) = \pi/4$, so that $\arg(1+i) = \pi/4 + 2\pi\mathbb{Z}$.
Therefore, with $z = 1+i$, $\boxed{\log z = \frac{1}{2}\log 2 + (\pi/4 + 2\pi n)i}$ for all integers $n \in \mathbb{Z}$
The principal value is $\boxed{\frac{1}{2}\log 2 + \pi i/4}$ (i.e., for $n = 0$). Here's the plot (not to scale):



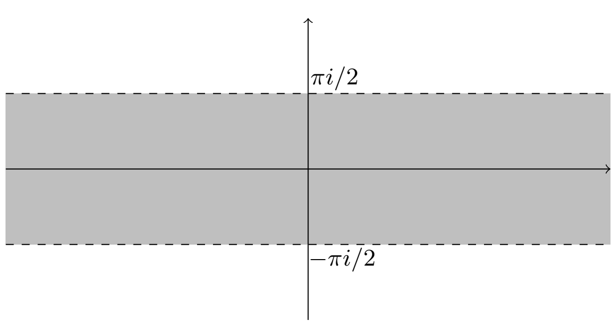
(d) Note that $\cos(\pi/3) = 1/2$ and $\sin(\pi/3) = \sqrt{3}/2$.
Thus, $|(1 + i\sqrt{3})/2| = 1$ and $\mathrm{Arg}((1 + i\sqrt{3})/2) = \pi/3$, so that $\arg i = \pi/3 + 2\pi\mathbb{Z}$.
Therefore, with $z = 1 + i\sqrt{3})/2$, $\boxed{\log z = (\pi/3 + 2\pi n)i}$ for all integers $n \in \mathbb{Z}$

The principal value is $\boxed{\pi i/3}$ (i.e., for $n = 0$). Here's the plot (not to scale):

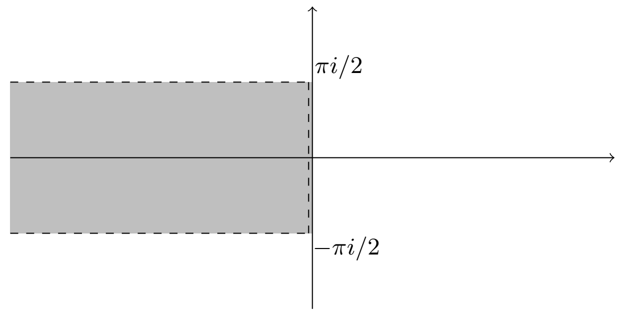


I.6, #2(a,b,d) Sketch the image under $w = \mathrm{Log}\, z$ of each of the following regions: (a): The right half-plane $\mathrm{Re}\, z > 0$. (b): The half-disk $|z| < 1$, $\mathrm{Re}\, z > 0$. (d): The slit annulus $\sqrt{e} < |z| < e^2$, $z \notin (-e^2, -\sqrt{e})$

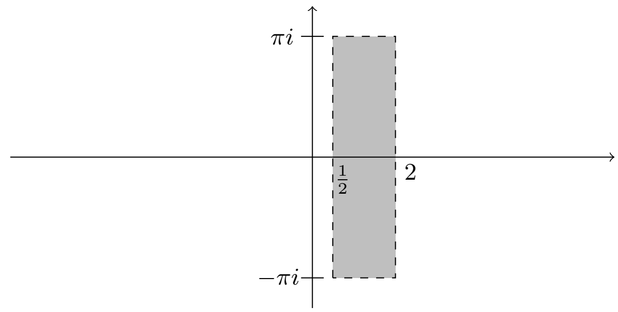
Solutions. (a) In polar, this half-plane is $-\pi/2 < \mathrm{Arg}\, z < \pi/2$. With no restrictions on the modulus of z, $\log |z|$ can be any real number. Thus, the image is the horizontal strip $-\pi/2 < \mathrm{Im}\, w < \pi/2$. Here's the picture:



(b) In polar, this half-disk is $0 < |z| < 1$ and $-\pi/2 < \mathrm{Arg}\, z < \pi/2$. Thus, $-\infty < \log |z| < 0$, and the image is the half-strip $-\pi/2 < \mathrm{Im}\, w < \pi/2$ with $\mathrm{Re}\, w < 0$. Here's the picture:



(d) In polar, this region is $\sqrt{e} < |z| < e^2$ and $-\pi < \operatorname{Arg} z < \pi$. Thus, $1/2 < \log|z| < 2$, and the image is the rectangle $-\pi < \operatorname{Im} w < \pi$ with $1/2 < \operatorname{Re} w < 2$. Here's the picture:

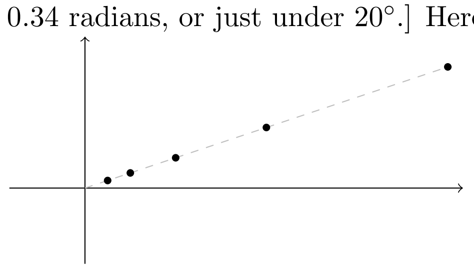


I.7, #1(a,b) Find and plot all values of: (a) $(1+i)^i$. (b) $(-i)^{1+i}$.

Solutions. By definition, $(1+i)^i = e^{i\log(1+i)}$.
By problem I.6 #1(c), $\log(1+i) = \frac{1}{2}\log 2 + (\pi/4 + 2\pi n)i$ for $n \in \mathbb{Z}$.
So $i\log(1+i) = -(\pi/4 + 2\pi n) + (\frac{1}{2}\log 2)i$ for $n \in \mathbb{Z}$. So

$$(1+i)^i = e^{i\log(1+i)} = e^{-\pi/4}e^{-2\pi n}e^{i\log\sqrt{2}} \quad \text{for } n \in \mathbb{Z},$$

which are complex numbers of argument $\log\sqrt{2}$, which is fairly small but positive. [You don't need to compute it, but FYI, it's about 0.34 radians, or just under 20°.] Here's the plot:

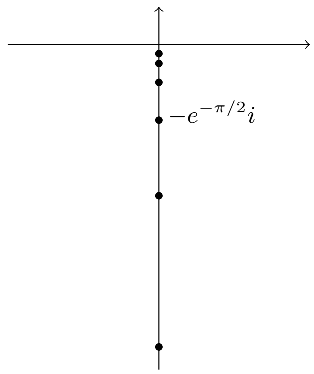


(The dots extend infinitely up and down the ray in the first quadrant, and the successive gaps between them increase by a factor of $e^{2\pi}$ as we head away from the origin.)

(b) By definition, $(-i)^{1+i} = e^{(1+i)\log(-i)}$.
We have $|-i| = 1$ and $\operatorname{Arg}(-i) = -\pi/2$, so $\log(-i) = (2\pi n - \pi/2)i$ for $n \in \mathbb{Z}$.
Therefore $(1+i)\log(-i) = (\pi/2 - \pi i/2) + (2\pi + 2\pi i)n$ for $n \in \mathbb{Z}$.
Note that $e^{\pi/2 - \pi i/2} = e^{\pi/2}e^{-\pi i/2} = -e^{\pi/2}i$ and $e^{2\pi+2\pi i} = e^{2\pi}e^{2\pi i} = e^{2\pi}$. Thus,

$$(-i)^{1+i} = e^{(1+i)\log(-i)} = \big(-e^{\pi/2}i\big)\big(e^{2\pi}\big)^n = -e^{\pi/2+2\pi n}i \quad \text{for } n \in \mathbb{Z},$$

which are purely imaginary complex numbers with negative imaginary part. Here's the plot:



(The dots extend infinitely up and down the negative imaginary axis, and the successive gaps between them increase by a factor of $e^{2\pi}$ as we head away from the origin.)

I.8, #1(a) Prove the identity $\cos(z+w) = \cos z \cos w - \sin z \sin w$.

Proof. For any $z, w \in \mathbb{C}$, we have $\cos(z+w) = \frac{1}{2}\Big(e^{i(z+w)} + e^{-i(z+w)}\Big) = \frac{1}{4}\Big(2e^{i(z+w)} + 2e^{-i(z+w)}\Big)$

$$= \frac{1}{4}\Big(e^{i(z+w)} + e^{i(z-w)} + e^{i(w-z)} + e^{-i(z+w)} + e^{i(z+w)} - e^{i(z-w)} - e^{i(w-z)} + e^{-i(z+w)}\Big)$$

$$= \frac{1}{4}\Big[\big(e^{iz} + e^{-iz}\big)\big(e^{iw} + e^{-iw}\big) + \big(e^{iz} - e^{-iz}\big)\big(e^{iw} - e^{-iw}\big)\Big]$$

$$= \frac{1}{2}\big(e^{iz} + e^{-iz}\big) \cdot \frac{1}{2}\big(e^{iw} + e^{-iw}\big) - \frac{1}{2i}\big(e^{iz} - e^{-iz}\big) \cdot \frac{1}{2i}\big(e^{iw} - e^{-iw}\big)$$

$$= \cos z \cos w - \sin z \sin w$$

QED

I.8, #4 Prove that $\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$ by proving that $\tan w = z$ if and only if $2iw$ is one of the values of $\log\left(\frac{1+iz}{1-iz}\right)$.

Proof. Given $z, w \in \mathbb{C}$, we proceed by a chain of if-and-only-if's, as follows:

$$\tan w = z \iff \frac{\sin w}{\cos w} = z \iff \frac{e^{iw} - e^{-iw}}{2i} \cdot \frac{2}{e^{iw} + e^{-iw}} = z \iff \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = iz$$

$$\iff \frac{e^{2iw} - 1}{e^{2iw} + 1} = iz \iff e^{2iw} - 1 = ize^{2iw} + iz \iff e^{2iw}(1 - iz) = 1 + iz$$

$$\iff e^{2iw} = \frac{1+iz}{1-iz} \iff 2iw = \log\left(\frac{1+iz}{1-iz}\right)$$

QED