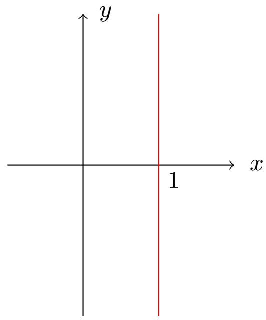
Math 345, Fall 2024

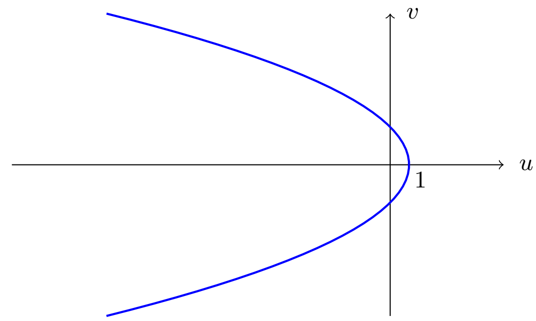
Solutions to Selected Homework Problems, HW #2

I.4, #1(b,f) Sketch each curve, and sketch its image under $w = z^2$. (b): $x = 1$. (f): $y = 1/x$ for $x \neq 0$.

Solutions. (b). Write $z = 1 + iy$ and $w = u + iv$. Then since $w = z^2$, we have $u + iv = (1 + iy)^2 = (1 - y^2) + 2yi$. That is, we have $u = 1 - y^2$ and $v = 2y$. Thus, $y = \frac{v}{2}$, so we have $u = 1 - \frac{v^2}{4}$.

So here are the graphs of $x = 1$ and its image $u = 1 - \frac{v^2}{4}$:



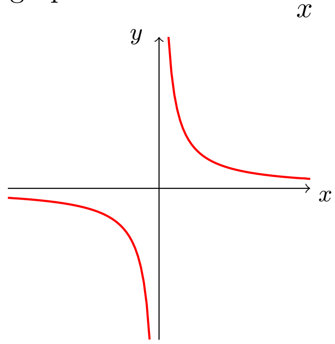


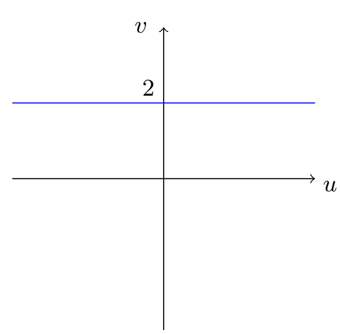
(f): Write $z = x + \frac{i}{x}$ with $x \neq 0$, and $w = u + iv$. Then since $w = z^2$, we have

$u + iv = \left(x + \frac{i}{x}\right)^2 = x^2 - \frac{1}{x^2} + 2i$. That is, we have $u = x^2 - x^{-2}$ and $v = 2$.

Note that the (real) function $u(x) = x^2 - x^{-2}$ is continuous on the domain $x \in (0, \infty)$ and has $\lim_{x\to\infty} u(x) = +\infty$ and $\lim_{x\to 0^+} u(x) = -\infty$. Thus, u must map $(0, \infty)$ onto the full real line $(-\infty, \infty)$. Similarly, u also maps $(-\infty, 0)$ onto the full real line $(-\infty, \infty)$. So the original curve $z = x + \frac{i}{x}$ with $x \neq 0$ covers the line $\text{Im}(w) = 2$ twice.

So here are the graphs of the $z = x + \frac{i}{x}$ and its image curve, $\text{Im}(w) = 2$.

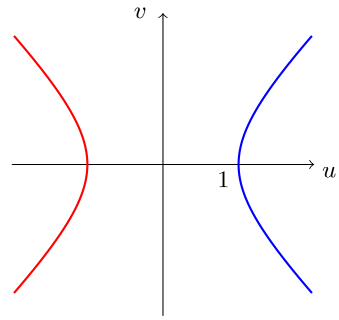




I.4, #2(b) Sketch the image of $x = 1$ under both branches of $w = \sqrt{z}$, in different colors.

Solution. The union of the two desired curves must be the set of all complex numbers $w = u + iv$ for which $\text{Re}(w^2) = 1$. Since $w^2 = (u^2 - v^2) + 2uvi$, this means the two curves together form the hyperbola $u^2 - v^2 = 1$.

This hyperbola consists of two separate arcs, one with $u \geq 1$ and one with $u \leq -1$. Since the image of the principal branch of $w = \sqrt{z}$ lies in the half-plane $\text{Re}\, w > 0$, and the other branch's image lies in the half-plane $\operatorname{Re} w < 0$, each of the arcs must be one branch. So here they are, with the image of the principal branch in blue and the other in red.



I.4, #3(a,b) (a): Give a brief description of the function $w = z^3$. (Describe what happens to a ray from the origin in the z-plane, and to a circle centered at the origin.) (b): Make branch cuts and define explicitly three branches of the inverse mapping.

Solution. (a): Writing $z = re^{i\theta}$ in polar, we have $w = r^3 e^{i(3\theta)}$, so that $|w| = |z|^3$ and $\arg w = 3 \arg z$.
Since $\arg w = 3 \arg z$, the ray from the original at angle θ_0 to the positive real axis is mapped to the ray at angle $3\theta_0$. (And because $|w| = |z|^3$, if z traverses this ray starting at the origin, then it starts moving slowing and then with increasing speed.)
Since $|w| = |z|^3$, the circle of radius r_0 centered at the origin is mapped to the circle of radius r^3. (And because $\arg w = 3 \arg z$, this mapping is 3-to-1; one loop around the z-circle traces out three loops around the w-circle.)

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(b): as in the discussion of $\sqrt{z}$ in Section I.4, make a branch cut in the w-plane by removing the branch cut $(-\infty, 0]$. So each $w \in \mathbb{C} \setminus (-\infty, 0]$ has a well defined Argument $\operatorname{Arg} w \in (-\pi, \pi)$.
For each $w \in \mathbb{C} \setminus (-\infty, 0]$, write $w = \rho e^{i\varphi}$, and define

$$f_1(w) = \rho^{1/3} e^{i\varphi/3}, \qquad f_2(w) = \rho^{1/3} e^{i(\varphi+2\pi)/3}, \qquad f_3(w) = \rho^{1/3} e^{i(\varphi+4\pi)/3}.$$

Then for each $j = 1, 2, 3$, we have

$$(f_j(w))^3 = \rho e * i\varphi = w,$$

so that each f_j is a branch of the inverse mapping to $w = z^3$.
And they are definitely different, because for any such $w = \rho e * i\varphi$, we have $-\pi/3 < \varphi/3 < \pi/3$, and hence

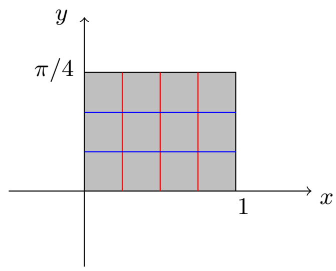
$$\operatorname{Arg}(f_1(w)) \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right), \qquad \operatorname{Arg}(f_2(w)) \in \left(\frac{\pi}{3}, \pi\right), \qquad \operatorname{Arg}(f_3(w)) \in \left(-\pi, -\frac{\pi}{3}\right),$$

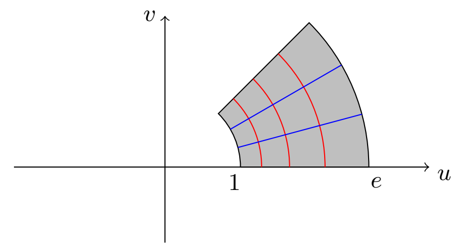
which describe three disjoint sectors of the z-plane.

I.5, #2(c) Sketch the region $0 < x < 1$, $0 < y < \pi/4$ and its imagine under $w = e^z$. Indicate the images of horizontal and vertical lines.

Solution. Write $z = x + iy$ and $w = se^{i\phi}$. Then $se^{i\phi} = w = e^z = e^x e^{iy}$, so we have $s = e^x$ and $\phi = y$ (with ϕ defined only up to adding integer multiples of 2π).
Since $0 < x < 1$, we have $1 < e^x < e$, so $1 < |w| < e$. In addition, each vertical line $\operatorname{Re}(z) = a$ becomes $s = e^a$, i.e., the circle $|w| = e^a$.
Since $0 < y < \pi/4$, we have $0 < \phi < \pi/4$, so $0 < \operatorname{Arg}(w) < \pi/4$. In addition, each horizontal line $\operatorname{Im}(z) = b$ becomes $\phi = b$, i.e., the ray $\operatorname{Arg}(w) = b$.

So here is the rectangle and its image:





I.5, #3. Show that $e^{\overline{z}} = \overline{e^z}$.

Proof. Given $z \in \mathbb{C}$, write $z = x + iy$. Then $\overline{z} = x - iy$, so
$e^{\overline{z}} = e^x e^{-iy} = e^x\big(\cos y + i\sin(-y)\big) = e^x\big(\cos y - i\sin y\big) = \overline{e^x\big(\cos y + i\sin y\big)} = \overline{e^x e^{iy}} = \overline{e^z}$ QED