Math 345, Fall 2024

Solutions to Homework #19

Problem 1. VII.1, #2(a). Calculate the residue of $f(z) = e^{1/z}$ at the isolated singularity at $z = 0$.

Solution. Substituting $1/z = z^{-1}$ in the usual power series for e^z gives

$f(z) = 1 + z^{-1} + \frac{1}{2!} \cdot z^{-2} + \frac{1}{3!} \cdot z^{-3} + \frac{1}{4!} \cdot z^{-4} + \cdots$

The coefficient of $z^{-1} = 1/z$ is $a_{-1} = 1$, so $\boxed{\text{Res}\,[f(z), 0] = 1}$

Problem 2. VII.1, #3(a,b). Use the Residue Theorem to evaluate the following integrals:

$$\text{(a)} \oint_{|z|=1} \frac{\sin z}{z^2}\, dz \qquad\qquad \text{(b)} \oint_{|z|=2} \frac{e^z}{z^2-1}\, dz$$

Solutions. (a): The only singularity of $f(z) = \dfrac{\sin z}{z^2}$ is at $z = 0$, which lies inside the contour.

The Laurent expansion at $z = 0$ is $f(z) = \dfrac{1}{z^2}\big(z + O(z^3)\big) = z^{-1} + O(z)$,

so the residue at $z = 0$ is $\text{Res}\,[f(z), 0] = 1$. Therefore, by the Residue Theorem,

$\oint_{|z|=1} \dfrac{\sin z}{z^2}\, dz = 2\pi i\, \text{Res}\,[f(z), 0] = \boxed{2\pi i}$

(b): $g(z) = \dfrac{e^z}{z^2-1}$ has poles at $z = \pm 1$, both of which are inside the contour, and no other singularities. More precisely, the numerator e^z is entire, while the denominator $z^2 - 1$ is entire with simple zeros at $z = \pm 1$. Moreover, the derivative of the denominator is $2z$.

Thus, by Rule 3, $\text{Res}\,[g(z), 1] = \left.\dfrac{e^z}{2z}\right|_{z=1} = \dfrac{e}{2}$, and $\text{Res}\,[g(z), -1] = \left.\dfrac{e^z}{2z}\right|_{z=-1} = \dfrac{e^{-1}}{-2} = -\dfrac{1}{2e}$.

Therefore, by the Residue Theorem, $\oint_{|z|=2} \dfrac{e^z}{z^2-1}\, dz = 2\pi i\left(\dfrac{e}{2} - \dfrac{1}{2e}\right) = \boxed{\pi i\left(e - \dfrac{1}{e}\right)}$

Problem 3. VII.2 #2. Use residue theory to show that for any any real constant $a > 0$, we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$

Solutions. Let $f(z) = (z^2 + a^2)^{-2} = (z + ia)^{-2}(z - ia)^{-2}$, which is analytic except at $z = \pm ia$, where it has double poles.

Only one of these poles, $z = ia$, lies inside the semicircular contour (for R large enough).

Note that $(z - ia)^2 f(z) = (z + ia)^{-2}$, so that $\dfrac{d}{dz}\big((z - ia)^2 f(z)\big) = -2(z + ia)^{-3}$.

Therefore, by Rule 2, we have $\text{Res}[f, ia] = \lim_{z \to ia}\big[-2(z + ia)^{-3}\big] = -2(2ia)^{-3} = -\dfrac{2}{(2ia)^3} = \dfrac{-i}{4a^3}$

For $R > a$, with Γ_R denoting the semicircular arc portion of the semicircular contour, we have

$|f(z)| = \dfrac{1}{|z^2 + a^2|^2} \geq \dfrac{1}{(|z|^2 - |a|^2)^2} = \dfrac{1}{(R^2 - a^2)}$ for z on Γ_R.

Since Γ_R has length πR, it follows from the ML-estimate that $0 \leq \left|\int_{\Gamma_R} f(z)\, dz\right| \leq \dfrac{\pi R}{(R^2 - a^2)}$

We have $\lim_{R\to\infty} \dfrac{\pi R}{(R^2-a^2)} = \lim_{R\to\infty} \dfrac{\pi R^{-3}}{(1-(a/R)^2)} = \dfrac{0}{(1-0)^2} = 0$

So by the squeeze law, $\lim_{R\to\infty} \left| \int_{\Gamma_R} f(z)\,dz \right| = 0$, and hence $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0$.

Therefore, with D_R denoting the filled-in semicircle enclosed by the semicircular contour,

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \lim_{R\to\infty} \int_{\partial D_R} f(z)\,dz = 2\pi i \operatorname{Res}\left[f, ia\right] = 2\pi i \cdot \left(\frac{-i}{4a^3}\right) = \frac{\pi}{2a^3}$$

as desired.

Note: It's OK to fast-forward the middle portion. That is, after using the ML-estimate to show that $\left| \int_{\Gamma_R} f(z)\,dz \right| \leq \dfrac{\pi R}{(R^2-a^2)}$, it's OK to jump to: So $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,dz = 0$ since $\lim_{R\to\infty} \dfrac{\pi R}{(R^2-a^2)} = 0$.

Problem 4. VII.2 #7. Use residue theory to show that for any real constant $a > 0$, we have

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{x^4+1}\,dx = \frac{\pi}{\sqrt{2}} e^{-a/\sqrt{2}} \left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}} \right).$$

Solution. Let $f(z) = \dfrac{e^{iaz}}{z^4+1}$, which is analytic except for simple poles at the four fourth roots of -1, which are at $\pm e^{i\pi/4}$ and $\pm e^{3i\pi/4}$.

Only two of these poles, $z = e^{i\pi/4} = \dfrac{1+i}{\sqrt{2}}$ and $z = e^{3i\pi/4} = \dfrac{-1+i}{\sqrt{2}}$, lie inside the semicircular contour.

The derivative of the denominator of f is $4z^3$. Therefore, by Rule 3, we have

$\operatorname{Res}\left[f(z), e^{i\pi/4}\right] = \left.\dfrac{e^{iaz}}{4z^3}\right|_{z=e^{i\pi/4}} = \dfrac{1}{4} e^{-3i\pi/4} \left(e^{a(-1+i)/\sqrt{2}} \right) = \dfrac{1}{4} e^{-a/\sqrt{2}} e^{i(a/\sqrt{2}-3\pi/4)}$, and

$\operatorname{Res}\left[f(z), e^{3i\pi/4}\right] = \left.\dfrac{e^{iaz}}{4z^3}\right|_{z=e^{3i\pi/4}} = \dfrac{1}{4} e^{-i\pi/4} \left(e^{a(-1-i)/\sqrt{2}} \right) = \dfrac{1}{4} e^{-a/\sqrt{2}} e^{i(-a/\sqrt{2}-\pi/4)}$.

Therefore,

$$\operatorname{Res}\left[f(z), e^{i\pi/4}\right] + \operatorname{Res}\left[f(z), e^{3i\pi/4}\right] = \frac{1}{4} e^{-a/\sqrt{2}} e^{-i\pi/2} \left[e^{i(a/\sqrt{2}-\pi/4} + e^{i(-a/\sqrt{2}+\pi/4} \right]$$

$$= \frac{-i}{4} e^{-a/\sqrt{2}} \left[2\cos\left(\frac{a}{\sqrt{2}} - \frac{\pi}{4} \right) \right] = \frac{-i}{2} e^{-a/\sqrt{2}} \left[\cos\left(\frac{a}{\sqrt{2}}\right) \cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{a}{\sqrt{2}}\right) \sin\left(\frac{\pi}{4}\right) \right]$$

$$= \frac{-i}{2\sqrt{2}} e^{-a/\sqrt{2}} \left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}} \right)$$

With Γ_R denoting the semicircular arc portion of the semicircular contour and $zx + iy$ on Γ_R, we have $|e^{iz}| = |e^{ix} e^{-y}| = e^{-y} \leq 1$.

Thus, for $R > 1$ and z on Γ_R, we have $|f(z)| = \dfrac{|e^{iz}|}{|z^4+1|} \leq \dfrac{1}{|z|^4-1} = \dfrac{1}{R^4-1}$.

By the ML-estimate, we have $0 \leq \left| \int_{\Gamma_R} f(z)\,dz \right| \leq \dfrac{\pi R}{R^4-1} \to 0$ as $R \to \infty$.

Therefore, $\int_{-\infty}^{\infty} f(z)\,dz = \lim_{R\to\infty} \int_{-R}^{R} f(z)\,dz = \lim_{R\to\infty} \int_{\partial D_R} f(z)\,dz = 2\pi i \left[\operatorname{Res}\left[f(z), e^{i\pi/4}\right] + \operatorname{Res}\left[f(z), e^{3i\pi/4}\right] \right]$

$$= \frac{2\pi i(-i)}{2\sqrt{2}} e^{-a/\sqrt{2}} \left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}} e^{-a/\sqrt{2}} \left(\cos\frac{a}{\sqrt{2}} + \sin\frac{a}{\sqrt{2}} \right)$$

So taking the real part — noting that $\operatorname{Re} f(x) = \dfrac{\cos ax}{x^4+1}$ for $x \in \mathbb{R}$ —

the original integral is $\dfrac{\pi}{\sqrt{2}} e^{-a/\sqrt{2}} \left(\cos\dfrac{a}{\sqrt{2}} + \sin\dfrac{a}{\sqrt{2}} \right)$, as desired.