Math 345, Fall 2024

Solutions to Homework #18

Problem 1. VI.4, #1(a,d). Find the partial fractions decompositions of the following functions.

(a) $\frac{1}{z^2 - z}$ (d) $\frac{1}{(z^2+1)^2}$

Solution. **(a)**: Observe that $f(z) = \frac{1}{z^2 - z} = \frac{1}{z(z-1)}$ has simple poles at $z = 0, 1$, and hence $f(z) - \left(\frac{a}{z} + \frac{b}{z-1}\right)$ must be entire for some $a, b \in \mathbb{C}$.

Moreover, since the degree of the numerator of f must be strictly less than that of the denominator, this entire function must go to 0 as $z \to \infty$, and hence it is constant and equal to 0.

OK, you can skip saying all of the above and just jump to the algebraic fact that we must have $f(z) = \frac{a}{z} + \frac{b}{z-1}$ for some $a, b \in \mathbb{C}$.

Summing the right side and comparing to f gives $a(z-1) + bz = 1$, i.e., $(a+b)z - a = 1$. Thus, we have $a = -1$ and hence $b = 1$.

[Alternatively, you could plug in $z = 0$ to get $-a = 1$, and $z = 1$ to get $b = 1$.]

That is, $\boxed{f(z) = -\frac{1}{z} + \frac{1}{z-1}}$

(d): Observe that $g(z) = \frac{1}{(z^2+1)^2} = \frac{1}{(z-i)^2(z+i)^2}$ has double poles at $z = i, -i$. Moreover, as in part (a), the numerator has smaller degree than the denominator, so g must equal the sum of its principal parts at these two poles. That is — and again, you can skip saying all of the above and just jump to the following — we have

$g(z) = \frac{a}{(z-i)} + \frac{b}{(z-i)^2} + \frac{c}{(z+i)} + \frac{d}{(z+i)^2}$ for some $a, b, c, d \in \mathbb{C}$.

Summing the right side and comparing to g gives

$a(z-i)(z+i)^2 + b(z+i)^2 + c(z-i)^2(z+i) + d(z-i)^2 = 1.$

Plugging in $z = i$ gives $b(2i)^2 = 1$, so that $-4b = 1$, and hence $b = -1/4$.

Similarly, plugging in $z = -i$ gives $d(-2i)^2 = 1$, so that $-4d = 1$, and hence $d = -1/4$.

Substituting $b = d = -1/4$ in the above equation and grouping terms gives

$(z-i)(z+i)\big[a(z+i) + c(z-i)\big] - \frac{1}{4}\big[(z^2 + 2iz - 1) + (z^2 - 2iz - 1)\big] = 1$

and hence

$(a+c)z(z^2+1) + i(a-c)(z^2+1) - \frac{1}{2}(z^2-1) = 1,$ i.e., $(a+c)z^3 + \left[i(a-c) - \frac{1}{2}\right]z^2 + (a+c)z + \left[i(a-c) - \frac{1}{2}\right] = 0$

Thus, we must have $a + c = 0$ and $2i(a-c) = 1$. With $c = -a$, the second of these equations becomes $4ia = 1$, so that $a = -i/4$ and hence $c = i/4$.

Putting it all together, we have $\boxed{g(z) = \frac{-i/4}{(z-i)} - \frac{1/4}{(z-i)^2} + \frac{i/4}{(z+i)} - \frac{1/4}{(z+i)^2}}$

Problem 2. VI.4, #2(a). Use the division algorithm to (help) obtain the partial fractions decomposition of the function $\dfrac{z^3+1}{z^2+1}$

Solution. Here's the long division:
$$\begin{array}{r} z \\ z^2+1 \overline{) \; z^3+1} \\ \underline{z^3+z} \\ -z+1 \end{array}$$

So our function is $f(z) = \dfrac{z^3+1}{z^2+1} = z + g(z)$ where $g(z) = \dfrac{-z+1}{z^2+1} = \dfrac{-z+1}{(z-i)(z+i)}$.

Because of the simple poles at $z = \pm i$, we may write $g(z) = \dfrac{a}{z-i} + \dfrac{b}{z+i}$ for some $a, b \in \mathbb{C}$.

Thus, $g(z) = \dfrac{a(z+i)+b(z-i)}{z^2+1}$, and hence $a(z+i)+b(z-i) = -z+1$.

Plugging in $z = i$ gives $2ia = 1-i$, so that $a = \dfrac{1}{2}(-1-i)$.

Plugging in $z = -i$ gives $-2ib = 1+i$, so that $a = \dfrac{1}{2}(-1+i)$.

Thus, the original function is $\boxed{z + \dfrac{(-1-i)/2}{z-i} + \dfrac{(-1+i)/2}{z+i}}$

Problem 3. VII.1 #1(a,b,c). Evaluate the following residues.

(a) $\operatorname{Res}\left[\dfrac{1}{z^2+4}, 2i\right]$ (b) $\operatorname{Res}\left[\dfrac{1}{z^2+4}, -2i\right]$ (c) $\operatorname{Res}\left[\dfrac{1}{z^5-1}, 1\right]$

Solutions. **(a)**: Since $g(z) = z^2+4 = (z+2i)(z-2i)$ has a simple zero at $z = 2i$, and since $g'(z) = 2z$, Rule 4 gives us $\operatorname{Res}\left[\dfrac{1}{z^2+4}, 2i\right] = \left.\dfrac{1}{2z}\right|_{z=2i} = \boxed{\dfrac{1}{4i}}$ (or $\dfrac{-i}{4}$, if you prefer).

(b): Since $g(z) = z^2+4 = (z+2i)(z-2i)$ has a simple zero at $z = -2i$, and since $g'(z) = 2z$, Rule 4 gives us $\operatorname{Res}\left[\dfrac{1}{z^2+4}, -2i\right] = \left.\dfrac{1}{2z}\right|_{z=-2i} = \boxed{-\dfrac{1}{4i}}$ (or $\dfrac{i}{4}$, if you prefer).

(c): Let $g(z) = z^5 - 1$, so that $g'(z) = 5z^4$. Since $g(1) = 0$ but $g'(1) = 5 \neq 0$, it follows that g has a simple zero at $z = 1$. Therefore, Rule 4 gives us $\operatorname{Res}\left[\dfrac{1}{z^5-1}, 1\right] = \left.\dfrac{1}{5z^4}\right|_{z=1} = \boxed{\dfrac{1}{5}}$

Problem 4. VII.1 #1(g,h). Evaluate the following residues.

(g) $\operatorname{Res}\left[\dfrac{z}{\operatorname{Log} z}, 1\right]$ (h) $\operatorname{Res}\left[\dfrac{e^z}{z^5}, 0\right]$

Solutions. **(g)**: Let $f(z) = z$ and $g(z) = \operatorname{Log} z$. Then $g'(z) = \dfrac{1}{z}$. We have $g(1) = 0$ and $g'(1) = 1 \neq 0$, so g has a simple zero at $z = 1$. Therefore, Rule 3 gives us $\operatorname{Res}\left[\dfrac{z}{\operatorname{Log} z}, 1\right] = \left.\dfrac{z}{1/z}\right|_{z=1} = \left.z^2\right|_{z=1} = \boxed{1}$

(h): Since this function has a pole of order 5 at $z = 0$, we can't use any of the rules. Fortunately, we can easily write down the Laurent series at $z = 0$.

We have $e^z = \sum_{k\geq 0} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + O(z^5)$,

and hence $\frac{e^z}{z^5} = z^{-5} + z^{-4} + \frac{1}{2}z^{-3} + \frac{1}{6}z^{-2} + \frac{1}{24}z^{-1} + O(z^0)$.

Reading off the coefficient of z^{-1}, then, we have $\operatorname{Res}\left[\frac{e^z}{z^5}, 0\right] = \boxed{\frac{1}{24}}$