Math 345, Fall 2024

Solutions to Homework #17

Problem 1. VI.1, #1(a). Find all possible Laurent expansions centered at 0 of $\dfrac{1}{z^2 - z}$

Solution. Call this function $f(z)$. Its denominator $z^2 - z = z(z-1)$ is zero at $z = 0, 1$, so that f is analytic on $\mathbb{C} \setminus \{0, 1\}$. Thus, there are two domains on which to consider Laurent decompositions: the punctured open disk $D_1 = \{0 < |z| < 1\}$ and the exterior domain $D_2 = \{|z| > 1\}$.

Write $f(z) = \dfrac{A}{z} + \dfrac{B}{z-1} = \dfrac{A(z-1) + Bz}{z(z-1)} = \dfrac{(A+B)z - A}{z^2 - z}$.

Thus, we must have $A = -1$ and $B = 1$, i.e., $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z}$.

On D_1, we have $|z| < 1$, so that $\dfrac{1}{z-1} = -\dfrac{1}{1-z} = -\sum_{k=0}^{\infty} z^k$.

Thus, the Laurent series on D_1 is $f(z) = -\dfrac{1}{z} - \sum_{k=0}^{\infty} z^k = \boxed{\sum_{k=-1}^{\infty} (-1) z^k} = -\dfrac{1}{z} - 1 - z - z^2 - z^3 - \cdots$.

On D_2, we have $|z| > 1$, so that $\dfrac{1}{z-1} = \dfrac{1}{z} \cdot \dfrac{1}{1 - \frac{1}{z}} = \dfrac{1}{z} \sum_{k=0}^{\infty} \dfrac{1}{z^k} = \sum_{k=1}^{\infty} z^{-k} = \sum_{k=-\infty}^{-1} z^k$.

Thus, the Laurent series on D_2 is $f(z) = -z^{-1} + \sum_{k=-\infty}^{-1} z^k = \boxed{\sum_{k=-\infty}^{-2} z^k} = \dfrac{1}{z^2} + \dfrac{1}{z^3} + \dfrac{1}{z^4} + \cdots$

Problem 2. VI.1, #1(c). Find all possible Laurent expansions centered at 0 of $\dfrac{1}{(z^2-1)(z^2-4)}$

Solution. Call this function $g(z)$. Its denominator is zero at $z = \pm 1, \pm 2$, so that g is analytic on $\mathbb{C} \setminus \{\pm 1, \pm 2\}$. Thus, there are three domains on which to consider Laurent decompositions: the open disk $D_1 = \{|z| < 1\}$, the open annulus $D_2 = \{1 < |z| < 2\}$, and the exterior domain $D_3 = \{|z| > 2\}$.

Write $g(z) = \dfrac{A}{z^2-1} + \dfrac{B}{z^2-4} = \dfrac{A(z^2-4) + B(z^2-1)}{(z^2-1)(z^2-4)} = \dfrac{(A+B)z^2 - (4A+B)}{(z^2-1)(z^2-4)}$.

Thus, we must have $A = -1/3$ and $B = 1/3$, i.e., $f(z) = \dfrac{-1/3}{z^2-1} + \dfrac{1/3}{z^2-4}$.

On D_1, we have $|z| < 1$, so that $\dfrac{-1/3}{z^2-1} = \dfrac{1/3}{1-z^2} = \dfrac{1}{3} \sum_{k=0}^{\infty} z^{2k}$,

and $\dfrac{1/3}{z^2-4} = \dfrac{-1/12}{1 - z^2/4} = -\dfrac{1}{12} \sum_{k=0}^{\infty} \dfrac{z^{2k}}{4^k}$,

Thus, the Laurent series on D_1 is $f(z) = \dfrac{1}{3} \sum_{k=0}^{\infty} z^{2k} - \dfrac{1}{12} \sum_{k=0}^{\infty} \dfrac{z^{2k}}{4^k} = \boxed{\sum_{k=0}^{\infty} \left(\dfrac{1}{3} - \dfrac{1}{12 \cdot 4^k} \right) z^{2k}}$

On D_2, we have $1 < |z| < 2$, so that the formula for $\dfrac{1/3}{z^2-4}$ is the same as for D_1, but now

$$\frac{-1/3}{z^2-1} = \frac{-1}{3z^2} \cdot \frac{1}{1 - \frac{1}{z^2}} = \frac{-1}{3z^2} \sum_{k=0}^{\infty} z^{-2k} = -\frac{1}{3} \sum_{k=0}^{\infty} z^{-2k-2} = -\frac{1}{3} \sum_{k=-\infty}^{-1} z^{2k}$$

Thus, the Laurent series on D_2 is $f(z) = \boxed{-\frac{1}{3}\sum_{k=-\infty}^{-1} z^{2k} - \frac{1}{12}\sum_{k=0}^{\infty}\frac{z^{2k}}{4^k}}$

On D_3 we have $|z| > 2$, so that the formula for $\frac{-1/3}{z^2-1}$ is the same as for D_2, but now

$$\frac{1/3}{z^2-4} = \frac{1}{3z^2}\cdot\frac{1}{1-\frac{4}{z^2}} = \frac{1}{3z^2}\sum_{k=0}^{\infty}4^k z^{-2k} = \frac{1}{3}\sum_{k=0}^{\infty}4^k z^{-2k-2} = \frac{1}{3}\sum_{k=-\infty}^{-1}\frac{z^{2k}}{4^{k+1}}$$

Thus, the Laurent series on D_3 is $f(z) = -\frac{1}{3}\sum_{k=-\infty}^{-1} z^{2k} + \frac{1}{3}\sum_{k=-\infty}^{-1}\frac{z^{2k}}{4^{k+1}} = \boxed{\sum_{k=-\infty}^{-1}\frac{1}{3}\left(\frac{1}{4^{k+1}}-1\right)z^{2k}}$

Problem 3. VI.2 #1(a). Find all of the isolated singularities (in $\mathbb{C}$, not at ∞) of $f(z) = \frac{z}{(z^2-1)^2}$. For each such singularity, determine whether it is removable, essential, or a pole. For each pole, determine its order, and find its principal part.

Solution. The denominator of $f(z) = \frac{z}{(z^2-1)^2} = \frac{z}{(z-1)^2(z+1)^2}$ is zero only at $z = \pm 1$, so f has singularities at those two points and nowhere else.

At $z = 1$, we have $f(z) = (z-1)^{-2}h_1(z)$, where $h_1(z) = \frac{z}{(z+1)^2}$ is analytic at $z = 1$ with $h_1(1) = 1/4 \neq 0$. Thus, f has a $\boxed{\text{pole of order 2 at } z = 1}$

We also have $h_1'(z) = \frac{(z+1)^2 - 2z(z+1)}{(z+1)^4} = \frac{1-z}{(z+1)^3}$, so that $h_1'(1) = 0$.

Thus, $h_1(z) = \frac{1}{4} + 0(z-1)^1 + O((z-1)^2)$, and hence the Laurent series expansion of f at $z = 1$ is $f(z) = \frac{1}{4}(z-1)^{-2} + O((z-1)^0)$. That is, $\boxed{\text{the principal part of } f \text{ at } z = 1 \text{ is } \frac{1}{4}(z-1)^{-2}}$

At $z = -1$, we have $f(z) = (z+1)^{-2}h_2(z)$, where $h_2(z) = \frac{z}{(z-1)^2}$ is analytic at $z = -1$ with $h_2(1) = -1/4 \neq 0$. Thus, f has a $\boxed{\text{pole of order 2 at } z = -1}$

We also have $h_2'(z) = \frac{(z-1)^2 - 2z(z-1)}{(z-1)^4} = \frac{-1-z}{(z-1)^3}$, so that $h_2'(-1) = 0$.

Thus, $h_2(z) = \frac{1}{4} + 0(z-1)^1 + O((z-1)^2)$, and hence the Laurent series expansion of f at $z = 1$ is $f(z) = -\frac{1}{4}(z+1)^{-2} + O((z+1)^0)$. That is, $\boxed{\text{the principal part of } f \text{ at } z = -1 \text{ is } \frac{-1}{4}(z+1)^{-2}}$

Note: Alternatively, one could do the partial fractions algebra to write $f(z) = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+1} + \frac{D}{(z+1)^2}$ and solve to get $A = C = 0$, $B = 1/4$, and $D = -1/4$.

Thus, after doing that annoying algebra, we get $f(z) = \frac{1/4}{(z-1)^2} - \frac{1/4}{(z+1)^2}$. From that, we can see that f has a pole at $z = 1$ of order 2 (because of the $(z-1)^{-2}$ term), with principal part $\frac{1/4}{(z-1)^2}$.

Similarly, f has a pole at $z = -1$ of order 2 with principal part $\frac{-1/4}{(z+1)^2}$.

Problem 4. VI.2 #1(c,e). Find all of the isolated singularities (in $\mathbb{C}$, not at ∞) of the following functions. For each such singularity, determine whether it is removable, essential, or a pole.

$$\text{(c)}\ \frac{e^{2z}-1}{z} \qquad\qquad \text{(e)}\ z^2 \sin\left(\frac{1}{z}\right)$$

Solutions. **(c)**: The denominator of $g(z) = \dfrac{e^{2z}-1}{z}$ is zero only at 0, so g has a singularity only at $z = 0$.

However, $e^{2z} = 1+2z+\dfrac{(2z)^2}{2!} = 1+2z+O(z^2)$, so that $e^{2z}-1 = 2z+O(z^2)$, and hence $g(z) = 2+O(z^1)$.

Thus, $\boxed{g \text{ has a removable singularity at } z = 0}$

(e): The function $f(z) = z^2 \sin\left(\dfrac{1}{z}\right)$ is analytic on $\mathbb{C} \setminus \{0\}$, so the only singularity is at $z = 0$.

Plugging $1/z$ into the standard power series for sine, we have

$$\sin\left(\frac{1}{z}\right) = z^{-1} - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!} + \cdots, \quad \text{so} \quad f(z) = z - \frac{z^{-1}}{3!} + \frac{z^{-3}}{5!} - \frac{z^{-5}}{7!} + \cdots.$$

Because infinitely many of the negative-power terms in this Laurent expansion are nonzero, $\boxed{f \text{ has an essential singularity at } z = 0}$

Problem 5. VI.2, #7. Let $z_0 \in \mathbb{C}$ be an isolated singularity of $f(z)$, and suppose that there is some $r > 0$ and integer $N \geq 1$ so that $(z - z_0)^N f(z)$ is bounded on $D(z_0, r)$. Prove that z_0 is either removable or else a pole of order at most N.

Proof. Let $g(z) = (z - z_0)^N f(z)$, which is bounded near z_0 by hypothesis. Therefore, by Riemann's Theorem on Removable Singularities, g has a removable singularity at z_0.

Hence, z_0 must be a removable singularity of g. Filling in the appropriate value for $g(z_0)$, we may assume that g is analytic on $D(z_0, r)$. Write $g(z) = \sum_{k \geq 0} b_k (z - z_0)^k$.

Thus, $f(z) = (z - z_0)^{-N} g(z) = b_0 (z - z_0)^{-N} + b_1 (z - z_0)^{-N+1} + b_2 (z - z_0)^{-N+2} + \cdots$ either has a pole of order at most N (if at least one of $b_0, \ldots, b_{N-1}$ is nonzero) or a removable singularity (if $b_0 = \cdots = b_{N-1} = 0$) at z_0 QED