Math 345, Fall 2024

Solutions to Homework #16

Problem 1. V.6, #2. Calculate the terms through order five (i.e., up to and including the z^5 term) of the power series expansion centered at $z=0$ of the function $f(z)=z/\sin z$.

Solution. We know $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + O(z^7)$.

Therefore, $\dfrac{\sin z}{z} = 1 - \dfrac{z^2}{3!} + \dfrac{z^4}{5!} + O(z^6)$. Hence,

$$\frac{z}{\sin z} = \frac{1}{(\sin z)/z} = \frac{1}{1-\frac{z^2}{3!}+\frac{z^4}{5!}+O(z^6)} = \frac{1}{1-\left(\frac{z^2}{3!}-\frac{z^4}{5!}+O(z^6)\right)}$$

$$= 1 + \left(\frac{z^2}{3!}-\frac{z^4}{5!}+O(z^6)\right) + \left(\frac{z^2}{3!}-\frac{z^4}{5!}+O(z^6)\right)^2 + \left(\frac{z^2}{3!}-\frac{z^4}{5!}+O(z^6)\right)^3 + \cdots$$

$$= 1 + \left(\frac{z^2}{3!}-\frac{z^4}{5!}\right) + \left(\frac{z^2}{3!}-\frac{z^4}{5!}\right)^2 + O(z^6) = 1 + \frac{z^2}{3!} - \frac{z^4}{5!} + \left(\frac{z^2}{3!}\right)^2 + O(z^6)$$

$$= 1 + \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + O(z^6) = \boxed{1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + O(z^6)}$$

Problem 2. V.6, #3. Write the power series expansion (centered at 0) of $f(z) = \dfrac{e^z}{1+z}$ as $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$.

(a) Prove that $a_0 = 1$, $a_1 = 0$, and $a_n = (-1)^n \left[\dfrac{1}{2!} - \dfrac{1}{3!} + \cdots + \dfrac{(-1)^n}{n!}\right]$ for all $n \geq 2$.

(b) Find the radius of convergence of this series (and of course prove your answer).

Solutions/Proofs. **(a)**:

(Method 1): We have $e^z = \displaystyle\sum_{k=0}^{\infty} \frac{1}{k!} z^k$ and $\dfrac{1}{1+z} = \displaystyle\sum_{m=0}^{\infty} (-1)^m z^m$, so by formula (6.1) on page 152 for the product of two series, we have $\displaystyle\sum_{n=0}^{\infty} a_n z^n = e^z \cdot \frac{1}{1+z} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{k!} \cdot (-1)^{n-k} \right) z^n$.

Thus, for each $n \geq 0$, we have $a_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!} \cdot (-1)^{n-k} = (-1)^n \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

For $n=0$, this formula gives $a_0 = \dfrac{1}{0!} = 1$, and for $n=1$, it gives $a_1 = (-1)\left(\dfrac{1}{0!} - \dfrac{1}{1!}\right) = 1 - 1 = 0$, as desired. For $n \geq 2$, we have:

$$a_n = (-1)^n \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\right) = (-1)^n \left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\right] \qquad \text{QED}$$

(a), **(Method 2)**: We have $(1+z)\displaystyle\sum_{n=0}^{\infty} a_n z^n = e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$.

The sum on the left is $\displaystyle\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_{n-1} z^n = a_0 + \sum_{n=1}^{\infty} (a_n + a_{n-1}) z^n$.

Thus, we have $a_0 + \displaystyle\sum_{n=1}^{\infty} (a_n + a_{n-1}) z^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$. Equivalently, $a_0 = \dfrac{1}{0!} = 1$, and for each $n \geq 1$, we have $a_n = \dfrac{1}{n!} - a_{n-1}$.

Hence, for $n = 1$, we have $a_1 = \dfrac{1}{1!} - a_0 = 1 - 1 = 0$.

We now prove the desired equality for $n \geq 2$ by induction.

For $n = 2$, our formula gives $a_2 = \dfrac{1}{2!} - a_1 = \dfrac{1}{2!} = (-1)^2\left[\dfrac{1}{2!}\right]$, verifying the desired equality for $n = 2$.

Assuming the equality holds for $n - 1$, our formula for n gives

$$a_n = \frac{1}{n!} - a_{n-1} = \frac{1}{n!} - (-1)^{n-1}\left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right]$$

$$= \frac{1}{n!} + (-1)^n\left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right] = (-1)^n\left[\frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right] \qquad \text{QED}$$

(b): The function $f(z)$ is analytic on $\mathbb{C} \setminus \{-1\}$, and because $\lim_{z\to -1} |f(z)| = \infty$ (since the denominator $1 + z$ approaches 0 and the numerator approaches $1/e \neq 0$), it follows that no extension of f can be analytic at -1.

Thus, the largest $R > 0$ such that f is analytic on $D(0, R)$ is $R = |-1-0| = 1$. Therefore, by the second Corollary on page 146, the radius of convergence of this power series is $\boxed{R = 1}$

Problem 3. V.7, #1(b,c,e). Find the zeros, and the orders of those zeros, of the following functions.

(b) $\dfrac{1}{z} + \dfrac{1}{z^5}$ **(c)** $z^2 \sin z$ **(e)** $\dfrac{\cos z - 1}{z}$

Solutions. **(a)**: Call this function $f(z)$, which we rewrite as $f(z) = \dfrac{z^4 + 1}{z^5}$.

Thus the zeros of f are the roots of $z^4 = -1$, i.e., $\boxed{\pm e^{i\pi/4} \text{ and } \pm e^{-i\pi/4}}$

[Note: there are other ways to write this, such as $\dfrac{1}{2}(\pm 1 \pm i)$ and as $e^{i(\pi/4 + j\pi/2)}$ for $j = 0, 1, 2, 3$, among other ways.]

We also have $f'(z) = -z^{-2} - 5z^{-6} = \dfrac{-(z^4 + 5)}{z^6}$. Thus, at each of the zeros z_j of f, since $z_j^4 = -1$, we have $f'(z_j) = \dfrac{-(-1 + 5)}{z_j^6} = \dfrac{-4}{z_j^6} \neq 0$. Hence, $\boxed{\text{each of the four roots of } f \text{ has order 1 as a zero of } f}$

(b): Let $g(z) = z^2 \sin z$, which we write as $g_1(z) \cdot g_2(z)$, where $g_1(z) = z^2$ and $g_2(z) = \sin z$.

Note that $g_1(z) = z^2$ has a zero only at $z = 0$, where the order of the zero is 2, since $g_1 = z^2 \cdot 1$.

Also observe that $g_2(z) = \sin z$ has zeros at $n\pi$ for $n \in \mathbb{Z}$. We have $g_2'(z) = \cos(z)$, so $g_2'(n\pi) = \pm 1 \neq 0$. Thus, each of these zeros of g_2 has order 1.

Recall (from page 155) that for any point z_0, the order of the zero of the product $g = g_1 g_2$ at z_0 is the sum of the orders of the zeros of g_1 and g_2 at z_0.

Thus, g has $\boxed{\text{a zero of order } 1 + 2 = 3 \text{ at } z = 0}$ and $\boxed{\text{a zero of order 1 at each } z = n\pi \text{ for } n \in \mathbb{Z} \setminus \{0\}}$

(c): Write $h(z) = \dfrac{\cos z - 1}{z} = \dfrac{h_1(z)}{h_2(z)}$, where $h_1(z) = \cos z - 1$ and $h_2(z) = z$.

Solving $h_1(z) = 0$ gives $z = 2\pi n$ for $n \in \mathbb{Z}$. Note that $h_1'(z) = -\sin z$ satisfies $h_1'(2\pi n) = 0$ for all $n \in \mathbb{Z}$, but $h_1''(z) = -\cos z$ has $h_1''(2\pi n) = -1 \neq 0$. Thus, h_1 has zeros of order 2 at each point $z = 2\pi n$ for $n \in \mathbb{Z}$, and no other zeros.

On the other hand, $h_2(z) = z$ has a zero of order 1 at $z = 0$, and no other zeros. Since h_1 has a zero of order 2 there, we may write $h_1(z) = z^2 H(z)$ with H analytic at 0 and $H(0) \neq 0$. Thus, $h(z) = zH(z)$, so that h has a zero of order 1 at $z = 0$.

Thus, h has $\boxed{\text{a zero of order 1 at } z = 0}$ and $\boxed{\text{a zero of order 2 at each } z = 2n\pi \text{ for } n \in \mathbb{Z} \setminus \{0\}}$

Problem 4. V.7, #6. Let f be analytic on a domain D, and let $z_0 \in D$. Suppose that $f^{(m)}(z_0) = 0$ for all $m \geq 1$. Prove that f is constant on D.

Proof. There is some $r > 0$ so that $D(z_0, r) \subseteq D$.

Let $c = f(z_0)$. Then the analytic function $g(z) = f(z) - c$ has $g(z_0)$ and $g^{(m)}(z_0) = 0$ for all $m \geq 1$. That is, $g^{(m)}(z_0) = 0$ for all $m \geq 0$.

Since g is equal to its Taylor series on the disk $D(z_0, r)$ (by the Taylor series Theorem on page 144), it follows that g is identically zero on $D(z_0, r)$.

In particular, z_0 is a non-isolated zero of g. Therefore, by the Theorem on page 156, g is identically zero on all of D. Thus, $f = g + c$ is identically equal to c on all of D. That is, f is constant. QED

Problem 5. V.7, #8. Let f and g be analytic functions on a domain D, and let $z_0 \in D$. Suppose that f has a zero of order $m \geq 0$ at z_0, and g has a zero of order $n \geq 0$ at z_0. Let k be the order of the zero of the function $f(z) + g(z)$ at z_0.

(a) Prove that $k \geq \min\{m, n\}$.

(b) If $m \neq n$, prove that $k = \min\{m, n\}$.

(c) Give an example to show that we *can* have $k > \min\{m, n\}$ in the case that $m = n$.

Proofs/Solutions. **(a)**: First, if $m = \infty$, then $f = 0$ and $k = n$, so we have $f + g = g$ has a zero of order $n = k$ at z_0, as desired. Similarly, if $n = \infty$, then by similar reasoning with the roles reversed, we have that $f + g = f$ has a zero of order $m = k$ at z_0.

For the remainder of the proof, then, we may assume that $m, n < \infty$.

Write $f(z) = (z - z_0)^m F(z)$ and $g(z) = (z - z_0)^n G(z)$, where F and G are analytic at z_0 with $F(z_0) \neq 0$ and $G(z_0) \neq 0$.

Then $f(z) + g(z) = (z - z_0)^\ell H(z)$ where $\ell = \min\{m, n\}$ and $H(z) = (z - z_0)^{m-\ell} F(z) + (z - z_0)^{n-\ell} G(z)$. Since $\ell \leq m$ and $\ell \leq n$, we have $m - \ell \geq 0$ and $n - \ell \geq 0$, and hence H is analytic at z_0.

Let $k' \geq 0$ be the order of the zero of H at z_0. (We have $k' \geq 0$ since H is analytic at z_0).

Then the order k of the zero of $f + g$ at z_0 is $k = \ell + k' \geq \ell$. QED

(b): Without loss of generality, assume $m < n$. Then $\min\{m, n\} = m$.
If $n = \infty$, then $g = 0$ and so $f + g = f$ has a zero of order $m = \min\{m, n\}$ at z_0, as desired.
So we may assume that $m < n < \infty$ for the rest of the proof.
Writing $f(z) = (z - z_0)^m F(z)$ and $g(z) = (z - z_0)^n G(z)$ as in part (a), we have

$f(z) + g(z) = (z - z_0)^m H(z)$ where $H(z) = F(z) + (z - z_0)^{n-m} G(z)$.

Thus, H is analytic at z_0, and $H(z_0) = F(z_0) + 0^{n-m} G(z_0) = F(z_0) \neq 0$, since $n - m > 0$.
Hence, the order k of the zero of $f + g$ at z_0 is $k = m = \min\{m, n\}$. QED

(c): There are many examples that would work, but the easiest one is probably this:
Let $z_0 = 0$, let $f(z) = z$, and let $g(z) = -z$. Then f and g both have zeros of order 1 at $z = 0$, but $f + g = 0$ has a zero of order ∞ there.

[If you want to cover all possible combinations of $m = n < k$, use $f = z^n$ and $g = -z^n$ if $k = \infty$ (so that $f + g = 0$), or use $f = z^n$ and $g = z^k - z^n$ (so that $f + g = z^k$) if $n < k < \infty$.]