Math 345, Fall 2024

Solutions to Homework #15

Problem 1. V.3, #7. Consider the series $\displaystyle\sum_{k=0}^{\infty}\big(2+(-1)^k\big)^k z^k$.

(a) Use the Cauchy-Hadamard formula to find the radius of convergence of this series.
(b) What happens when the ratio test is applied?
(c) Explicitly evaluate the sum of the series.

Solution/Proof. Let $a_k = \big(2+(-1)^k\big)^k$. That is, $a_k = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ 3^k & \text{if } k \text{ is even.} \end{cases}$

Part (a): We compute

$$\begin{aligned}\limsup_{k\to\infty} \sqrt[k]{|a_k|} &= \lim_{n\to\infty} \sup\Big(\big\{\sqrt[k]{1}\big|k \geq n \text{ odd}\big\} \cup \big\{\sqrt[k]{3^k}\big|k\geq n \text{ even}\big\}\Big) \\ &= \lim_{n\to\infty} \max\{1,3\} = \lim_{n\to\infty} 3 = 3.\end{aligned}$$

Hence, by Cauchy-Hadamard, the radius of convergence is $1/3$.

Part (b): If we apply the ratio test, we have

$$\left|\frac{a_k}{a_{k+1}}\right| = \frac{1}{3^{k+1}} \text{ for } k \text{ odd}, \qquad \left|\frac{a_k}{a_{k+1}}\right| = 3^k \text{ for } k \text{ even}.$$

Thus, $\displaystyle\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|$ diverges, since the odd terms go to 0, and the even terms go to ∞. That is, the ratio test is inconclusive.

Part (c): To sum the series, sum the even and odd terms separately. That is, writing $k=2n$ for the even terms, and $k=2n+1$ for the odd terms, the Geometric Series Test yields

$$\sum_{k=0}^{\infty} a_k z^k = \sum_{n=0}^{\infty} 3^{2n} z^{2n} + \sum_{n=0}^{\infty} z^{2n+1} = \boxed{\frac{1}{1-9z^2} + \frac{z}{1-z^2}}$$

Problem 2. V.4 #1(a,b,d). Find the radius of convergence of the power series for each of the following functions, expanding about the indicated point.

(a) $\dfrac{1}{z-1}$, about $z=i$ (b) $\dfrac{1}{\cos z}$, about $z=0$ (d) $\operatorname{Log} z$, about $z=1+2i$

Solutions. **(a)**: Note that $|1-i|=\sqrt{2}$. Therefore, $f(z)=\dfrac{1}{z-1}$ is analytic on $D(i,\sqrt{2})$ but blows up at the point $z=1$ at distance $\sqrt{2}$ from i.
Therefore, by the second Corollary on page 146, the radius of convergence is $\boxed{\sqrt{2}}$

— — — — — — —

(b): The function $\cos z$ has zeros at all odd multiples of $\pi/2$ and nowhere else.
Therefore, $f(z)=\dfrac{1}{\cos z}$ is analytic on $D(0,\pi/2)$ but blows up at the points $z=\pm\pi/2$ at distance $\pi/2$ from 0.
Therefore, by the second Corollary on page 146, the radius of convergence is $\boxed{\pi/2}$

— — — — — — —

(c): The function $\operatorname{Log} z$ fails to be analytic at $z=0$, which is at distance $|1+2i|=\sqrt{5}$ from $1+2i$.

At the same time, $\operatorname{Log} z$ is analytic on a region (e.g., the slit plane) containing the disk $D(1+2i, \sqrt{5})$. Therefore, by the second Corollary on page 146, the radius of convergence is $\boxed{\sqrt{5}}$

Problem 3. V.4 #2. Prove that the radius of convergence of the power series expansion of $\dfrac{z^2-1}{z^3-1}$ about $z=2$ is $R=\sqrt{7}$.

Proof. Let $f(z)=\dfrac{z^2-1}{z^3-1}$, and then, cancelling a factor of $z-1$ from both numerator and denominator, let $g(z)=\dfrac{z+1}{z^2+z+1}$.

Note that g is analytic on a larger domain than f is — g is defined at 1, in particular — but there cannot an analytic extension of either function that is analytic at either of the roots of z^2+z+1, since the value of $|g(z)|$ blows up to ∞ at those points.

The roots of z^2+z+1 are $z=\dfrac{-1\pm\sqrt{1-4}}{2}=-\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$, by the quadratic formula. Both of these points are at distance $\sqrt{\left(2+\dfrac{1}{2}\right)^2+\left(\dfrac{\sqrt{3}}{2}\right)^2}=\sqrt{\dfrac{25}{4}+\dfrac{3}{4}}=\sqrt{7}$ from $z=2$.

Thus, g is analytic on $D(2,\sqrt{7})$, but no extension of g is analytic on any larger open disk.
Therefore, by the second Corollary on page 146, the radius of convergence is $\sqrt{7}$. QED

Problem 4. V.4 #3. Find the power series expansion of $\operatorname{Log} z$ about the point $z=i-2$. Working directly from this series, prove that its radius of convergence is $R=\sqrt{5}$. Explain why this does not contradict the discontinuity of $\operatorname{Log} z$ at $z=-2$.

Solution/Proof. Let $f(z)=\operatorname{Log} z$. Then $f'(z)=z^{-1}$, so that $f''(z)=-z^{-2}$ and $f'''(z)=2z^{-3}$, and in general, $f^{(k)}(z)=(-1)^{k-1}(k-1)!z^{-k}$.

Thus, $f(i-2)=\operatorname{Log}(i-2)$, but for $k\geq 1$, we have $f^{(k)}(i-2)=(-1)^{k-1}(k-1)!(i-2)^k$.

Therefore, by the Taylor series formula (4.1), (4.2) on page 144, we have

$$f(z)=\operatorname{Log}(i-2)+\sum_{k\geq 1}a_k(z-(i-2))^k, \text{ where } a_k=\frac{(-1)^{k-1}(k-1)!(i-2)^{-k}}{k!}=\frac{(-1)^k}{k(i-2)^k}.$$

Applying the ratio test (from page 141), the radius of convergence is

$$R=\lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|=\lim_{k\to\infty}\frac{k+1}{k}\cdot|i-2|=|i-2|=\sqrt{5}, \text{ as claimed.}$$

This does not contract the discontinuity of $\operatorname{Log} z$ at -2 (and along the slit $(-\infty,0]$), because there is a different analytic branch of $\log z$ — say, with $\arg z\in(0,2\pi)$, which has branch cut along $[0,\infty)$ — for which the disk $D(i-2,\sqrt{5})$ is contained in the domain.

Problem 5. V.4, #12. Let $f(z)$ be an analytic function with power series expansion $\sum a_n z^n$.
If f is an even function (i.e., $f(-z)=f(z)$), prove that $a_n=0$ for all n odd.
If f is an odd function (i.e., $f(-z)=-f(z)$), prove that $a_n=0$ for all n even.

Proof. Define $g(z)=f(-z)$. Then by the Chain Rule, we have $g'(z)=-f'(-z)$ and $g''(z)=f''(-z)$; proceeding inductively, we have $g^{(k)}(z)=(-1)^k f^{(k)}(z)$.

Thus, for any $k\geq 0$, we have $g^{(k)}(0)=(-1)^k f^{(k)}(0)$.

Since g is also analytic, we may write $g(z) = \sum b_n z^n$. By the Theorem on page 144, we have $a_n = \dfrac{f^{(n)}(0)}{n!}$ and $b_n = \dfrac{g^{(n)}(0)}{n!}$. It follows that $b_n = (-1)^n a_n$ for all $n \geq 0$.

Case 1: f is even. Then $g = f$, and hence, by the uniqueness of power series expansions (via the first Corollary on page 146), we have $b_n = a_n$ for all n.

On the other hand, by the above, for any n odd, we have $b_n = (-1)^n a_n = -a_n$. That is, $a_n = -a_n$, and hence a_n=0, as desired.

Case 2: f is odd. Then $g = -f$, and hence, by the uniqueness of power series expansions (via the first Corollary on page 146), we have $b_n = -a_n$ for all n.

On the other hand, by the above, for any n even, we have $b_n = (-1)^n a_n = a_n$. That is, $a_n = -a_n$, and hence a_n=0, as desired. QED