Math 345, Fall 2024

Solutions to Homework #14

Problem 1. V.2, #7. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence of complex numbers. For any $\varepsilon > 0$, prove that the series $\sum_{n=1}^{\infty} a_n n^{-z}$ converges uniformly on the (closed) half-plane $\operatorname{Re} z \geq 1+\varepsilon$.

Proof. By hypothesis, there is some $M > 0$ such that $|a_n| \leq M$ for all $n \geq 0$. Given $\varepsilon > 0$:

Claim. For each $n \geq 1$ and each z with $\operatorname{Re} z \geq 1+\varepsilon$, we have $|a_n n^{-z}| \leq M/n^{1+\varepsilon}$.

Proof of Claim. Given $n \geq 1$ and $z = x+iy$ with $x \geq 1+\varepsilon$, we have

$$\left|a_n n^{-z}\right| = \left|a_n e^{-z \operatorname{Log} n}\right| = |a_n||e^{-x \log n}||e^{-iy \log n}| = |a_n||n^{-x}| \leq M n^{-x} \leq \frac{M}{n^{1+\varepsilon}}. \qquad \text{QED Claim}$$

In addition, we have $\sum_{n\geq 1} \frac{M}{n^{1+\varepsilon}} = M \sum_{n\geq 1} \frac{1}{n^{1+\varepsilon}}$ converges by the p-test (from Math 121). Thus, by the Weierstrass M-test, the original series converges uniformly on $\{z \in \mathbb{C} : \operatorname{Re} z \geq 1+\varepsilon\}$. QED

Problem 2. V.2 #8. Prove that $\sum_{k\geq 1} \frac{z^k}{k^2}$ converges uniformly on the disk $|z| < 1$.

Proof. Define $M_k = \frac{1}{k^2}$ for each $k \geq 1$. Then for all $z \in D(0,1)$ and all $k \geq 1$, we have

$$\left|\frac{z^k}{k^2}\right| = \frac{|z|^k}{k^2} \leq \frac{1^k}{k^2} = M_k.$$

We also know that $\sum M_k = \sum \frac{1}{k^2}$ converges by the p-test. Therefore, the original series converges uniformly on $D(0,1)$ by the M-test. QED

Problem 3. V.3, #1(a,b,d). Find the radius of convergence of each of the following power series.

(a) $\sum_{k=0}^{\infty} 2^k z^k$ (b) $\sum_{k=0}^{\infty} \frac{k}{6^k} z^k$ (d) $\sum_{k=0}^{\infty} \frac{3^k z^k}{4^k + 5^k}$

Solutions. **(a), Method 1**: By definition, the radius of convergence is $\sup \mathcal{R}$, where
$\mathcal{R} = \{r \geq 0 \,|\, \{|2^k| r^k\}_k \text{ is bounded}\}$.
The sequence here is $\{(2r)^k\}_k$, which, being a geometric sequence, is bounded if and only if its common ratio $2r$ has absolute value < 1. That is, $\mathcal{R} = \{r \geq 0 \,|\, 2r < 1\} = [0, 1/2)$, which has supremum $R = 1/2$. That is, the radius of convergence is $\boxed{\frac{1}{2}}$

(a), Method 2: Applying the version of the Ratio Test on page 141, the radius of convergence is

$$\lim_{k\to\infty} \left|\frac{2^k}{2^{k+1}}\right| = \lim_{k\to\infty} \frac{1}{2} = \boxed{\frac{1}{2}}$$

(b), Method 1: By definition, the radius of convergence is $\sup \mathcal{R}$, where
$\mathcal{R} = \left\{r \geq 0 \,\middle|\, \left\{\left|\frac{k}{6^k}\right| r^k\right\}_k \text{ is bounded}\right\}$. The sequence here is $\{k(r/6)^k\}_k$. We claim that $\mathcal{R} = [0, 6)$.

We prove the forward inclusion of this claim by contrapositive: suppose $r \geq 6$. Then $k(r/6)^k \geq k$, so that $\{k(r/6)^k\}_k$ is unbounded, and hence $r \notin \mathcal{R}$, as desired.
To prove the reverse inclusion of the claim, consider arbitrary $r \in [0, 6)$.
Then writing $a = \log(6/r) > 0$, we have $(6/r)^k = e^{ak}$, and so $\lim_{k\to\infty} \frac{k}{(6/r)^k} = \lim_{t\to\infty} \frac{t}{e^{at}} = \lim_{t\to\infty} \frac{1}{ae^{at}} = 0$, by L'Hôpital's rule, since the second limit is of the indeterminate form ∞/∞. Therefore, since convergent sequences are bounded (see the Theorem on page 34, for example), it follows that $r \in \mathcal{R}$, proving our claim. Hence, $\sup \mathcal{R} = 6$. So the radius of convergence is $\boxed{6}$

(b), Method 2: Applying the version of the Ratio Test on page 141, the radius of convergence is

$$\lim_{k\to\infty} \left| \frac{\frac{k}{6^k}}{\frac{k+1}{6^{k+1}}} \right| = \lim_{k\to\infty} \frac{k}{k+1} \cdot \frac{6^{k+1}}{6^k} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{k}} \cdot 6 = \boxed{6}$$

(d), Method 1: By definition, the radius of convergence is $\sup \mathcal{R}$, where $\mathcal{R} = \left\{ r \geq 0 \,\middle|\, \left\{ \left| \frac{3^k}{4^k + 5^k} \right| r^k \right\}_k \text{ is bounded} \right\}$. The sequence here is $\left\{ \frac{(3r/5)^k}{(4/5)^k + 1} \right\}_k$. We claim that $\mathcal{R} = [0, 5/3]$.

We prove the forward inclusion of this claim by contrapositive: suppose $r > 5/3$, and hence that $3r/5 > 1$. Then $\frac{(3r/5)^k}{(4/5)^k + 1} \geq \frac{1}{2}\left(\frac{3r}{5}\right)^k$ is unbounded, so that $r \notin \mathcal{R}$, as deesired.
To prove the reverse inclusion of the claim, consider arbitrary $r \in [0, 5/3)$. Then $3r/5 < 1$. So $\frac{(3r/5)^k}{(4/5)^k + 1} \leq \left(\frac{3r}{5}\right)^k < 1$ is bounded. Therefore, $r \in \mathcal{R}$, proving our claim.

Hence, $\sup \mathcal{R} = 5/3$. So the radius of convergence is $\boxed{\frac{5}{3}}$

(d), Method 2: Applying the version of the Ratio Test on page 141, the radius of convergence is

$$\lim_{k\to\infty} \left| \frac{\frac{3^k}{4^k + 5^k}}{\frac{3^{k+1}}{4^{k+1} + 5^{k+1}}} \right| = \lim_{k\to\infty} \frac{3^k}{3^{k+1}} \cdot \frac{4^{k+1} + 5^{k+1}}{4^k + 5^k} = \lim_{k\to\infty} \frac{1}{3} \cdot \frac{4 \cdot (\frac{4}{5})^k + 5}{(\frac{4}{5})^k + 1} = \frac{1}{3} \cdot \frac{0+5}{0+1} = \boxed{\frac{5}{3}}$$

Problem 4. V.3, #5(a). What function is represented by the power series $\sum_{k=1}^{\infty} kz^k$?

Solution. Recall that $(1-z)^{-1} = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k$; call this function $g(z)$.
Then differentiating the power series, we have

$$g'(z) = \sum_{k=0}^{\infty} kz^{k-1} = \sum_{k=1}^{\infty} kz^{k-1},$$

where the index change at the end was simply by the fact that the $k = 0$ term is already 0. On the other hand, the Chain Rule yields $g'(z) = -(1-z)^{-2} \cdot (-1) = \frac{1}{(1-z)^2}$.

Thus, the original power series is $\displaystyle\sum_{k=1}^{\infty} kz^k = zg'(z) = \boxed{\frac{z}{(1-z)^2}}$

Problem 5. V.3, #6. Show that a power series $\sum a_k z^k$, its differentiated series $\sum k a_k z^{k-1}$, and its integrated series $\sum \frac{a_k}{k+1} z^{k+1}$ all have the same radius of convergence.

Proof. Let $\mathcal{R}_1 = \{r \in [0,\infty) | \{|a_k| r^k\}_{k\geq 0} \text{ is bounded}\}$, and let
$\mathcal{R}_2 = \{r \in [0,\infty) | \{|ka_k| r^{k-1}\}_{k\geq 1} \text{ is bounded}\}$. We will show that $\sup \mathcal{R}_1 = \sup \mathcal{R}_2$.
First, given $r \in \mathcal{R}_2$, let M_r be a bound for the sequence $\{|ka_k| r^{k-1}\}_{k\geq 1}$. Then for each $k \geq 1$, we have

$$|a_k| r^k = r\big(|a_k| r^{k-1}\big) \leq r\big(|ka_k| r^{k-1}\big) \leq rM_r.$$

Thus, for all $k \geq 0$, we have $|a_k| r^k \leq \max\{|a_0|, rM_r\}$. Hence, $\{|a_k| r^k\}_{k\geq 0}$ is bounded; that is, $r \in \mathcal{R}_1$. We have shown that $\mathcal{R}_2 \subseteq \mathcal{R}_1$. Therefore, any upper bound for $\mathcal{R}_1$ is also an upper bound for $\mathcal{R}_2$. Thus, $\sup \mathcal{R}_1 \geq \sup \mathcal{R}_2$.
Second, given $r \in \mathcal{R}_1$, we will now show that $[0,r) \subseteq \mathcal{R}_2$. This is trivial if $r = 0$, so we assume $r > 0$. Let N_r be a bound for the sequence $\{|a_k| r^k\}_{k\geq 0}$. Given $s \in [0,r)$, we have $\lim_{k\to\infty} k(s/r)^{k-1} = 0$, and hence there is some $B \geq 0$ such that $k(s/r)^k \leq B$ for all $k \geq 0$. Thus, for all $k \geq 1$, we have

$$|ka_k| s^{k-1} = \left[k\left(\frac{s}{r}\right)^{k-1}\right]\big(|a_k| r^k\big)\left(\frac{1}{r}\right) \leq \frac{BN_r}{r}.$$

Therefore, the sequence $\{|ka_k| s^{k-1}\}_{k\geq 1}$ is bounded (by BN_r/r). That is, $s \in \mathcal{R}_2$, as desired.
Given any upper bound C for $\mathcal{R}_2$, then for every $r \in \mathcal{R}_1$, we have $[0,r) \subseteq \mathcal{R}_2$, and hence $C \geq r$. Thus, C is also an upper bound for $\mathcal{R}_1$. Hence, $\sup \mathcal{R}_1 \leq \sup \mathcal{R}_2$. By our previous inequality, $\sup \mathcal{R}_1 = \sup \mathcal{R}_2$.
Since $\sup \mathcal{R}_1$ is the radius of convergence of $\sum a_k z^k$ and $\sup \mathcal{R}_2$ is the radius of convergence of $\sum k a_k z^{k-1}$, it follows that the two series have the same radius of convergence.
Now consider the series $\sum \frac{a_k}{k+1} z^{k+1}$ in place of $\sum a_k z^k$. By what we have just proven, the derivative of the series $\sum \frac{a_k}{k+1} z^{k+1}$ has the same radius of convergence as $\sum \frac{a_k}{k+1} z^{k+1}$ itself. That is, $\sum a_k z^k$ and $\sum \frac{a_k}{k+1} z^{k+1}$ have the same radius of convergence. QED

Side note #1: We cannot use the ratio test, because there are some series where $\lim_{k\to\infty} |a_k/a_{k+1}|$ diverges. (See, for example, Exercise V.3 #7, which is on HW #15).

Side note #2: An alternate proof strategy would be to use the Cauchy-Hadamard formula, which reduces to proving that

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{k\to\infty} \sqrt[k-1]{|ka_k|}.$$

(This can be made slightly easier by first multiplying the differentiated series by z — which does not change the region of convergence, and hence does not change the radius of convergence — so that we have to show $\limsup_{k\to\infty} \sqrt[k]{|a_k|} = \limsup_{k\to\infty} \sqrt[k]{|ka_k|}$. But that's just a slight improvement.)
Unfortunately, we haven't proven any rules for manipulating limsup's. (And while there *are* some rules for manipulating limsups, they are not as easy as the limit laws. For example, in general, $\limsup(a_k + b_k)$ does *not* equal $\limsup a_k + \limsup b_k$.) So if you proceed in this fashion, you have to work from the definition of $\limsup$. And that ends up taking at least as much time and effort as the proof I gave above.